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CONTENTS

	PAGE
Augustine, K. O. (Konnully, A): A note on Cauchy's theorem	82— 83
————————: Pedal conic	180—181
Bagchi, N.: An alternate method of finding out the square root of a number	177—180
Balachandran, K.: Power series for circular functions ...	174—177
Chidambaram, S.: On a particular inscribed rectangle of a conic	143—147
Chowla S. and Auluck F. C.: An approximation with exp. x	75— 77
————————: On Weirstrass approximation theorem	78— 79
Gupta, H.: A problem in combinations	131—132
Gunjukar K. R. and Oza. H. P.: Joule's minimum dissipation theorem	40
Kesava Menon, P.: A generalisation of the circular and hyperbolic functions	112—117
———————— A theorem on Congruence	156—158
Krishnaswamy Ayyangar, A. A: Lecture notes ...	83— 88
————————: Rene Descartes	101—108
————————: The role of unit partial quotients in some continued fractions	159—166
Kosambi, D. D.: A note on frequency distribution in series...	151—155
Levi, F. W.: A problem on rigid notion	1— 10
Madhava Rao, B. S.: Generalised geometry and physical theories	11— 34
Miller, G. A.: Enumeration of finite groups	109—111
Narasimhamurty, V.: On demlo numbers	34— 37
Narasinga Rao, A.: Studies in circle geometry ...	53— 72
————————: A fallacy in deducing the attraction of a plane from that of a spherical shell ...	182—183
Neville, E. H.: A proof of an elementary theorem on complex numbers	173
Pillai, S. S.: On the converse of Fermat's Theorem ...	132—133
Rajagopal, C. T.: On Abel's divergence test for series of positive terms	118—123
Rangaswami, K.: A note on the harmonic envelope ...	37— 39
————————: Remarks on Mr. Augustine's note ...	181—182
Sambasiva Rao, K.: Lecture Notes	89

Sastry, B. S.: The limiting points of the set of the upper and lower sums in Riemann integration ... 148—150

Sastry, S.: On the sum of $\sum x^{12n}/(12n)!$ ... ... 133—134

Sitaraman, B.: On correlation constants in mingled records 73— 75

Sperling, Maurice.: On the number π ... ... 41— 43

Srinivasan, G.: The infinitesimal flow in the inversive plane ... ... 167—170

Srinivasiengar, C. N.: The line of umbilics ... ... 171—172

Srinivasa Sastry, H.: On the function defined by the relation $\phi_r(x) = x\phi_{r-1}(x)$ ... ... 127—130

Vaidyanatha Sastry, M.: Heliacal rising of the moon ... 80— 81

Announcements and News 48, 95, 140

Solutions to Questions ... 49—51, 96—99, 134—139, 184—186

Questions for Solution ... ... 51—52, 100, 141—142

Gleanings 10, 43, 52, 130, 142, 155, 158, 166, 170

Books received for Review 47

Reviews:

Weyrich: *Die Zylinderfunktionen and ihre Anwendungen* 44

Vasilesco, F: *La notion de capacite* ... ... 44— 46

————: *La notion de point irregulier*... ... 44— 46

Mandelbrojt: *La reqularisation des Fonctions* ... 47

Colloque consacre a la Theorie des Probabilites: Actualites Scientifiques Nos. 734, 735, 736, 737, 738 and 739: 90— 95

Author Index of Questions and Solutions

(The numbers refer to pages)

Auluck, J. C.
Solutions 185, 186.

Balachandran, K.:
Question 141.
Solution 185.

Court, N. A.:
Question 100.

Hariharan, P.
Solutions 185², 186.

Iyengar, K. S. K.:
Question 51, 135.
Solution 136.

Kaprekar, D. R.
Question 141.

Kesava Menon, P.
Questions 100², 141, 185.

Krishnamoorthi, C. R.
Solution 185.

Krishnaswamy Ayyangar, A. A.
Questions 97, 100, 142.

Krishnaswamy, M. N.
Question 52.

Manickavasagam, P. S. J.
Solution 186.

Moessner, A.
Question 97.

Narasimhamurthy, N. K.
Solution 49.

Narasimhamurthy, V. K.
Solution 135.

Palkar, A. E.
Questions, 100, 185.

Radhakrishna Rao, C.
Solution 186.

Raghavachari, T. K.
Solution 50, 135.

Ramanathan, K. G.
Question

Ramanujam, R.
Question 96.
Solution 96.

Rathie, C. B.
Solution 98.

Regan, F.
Question 50.

Sen, D. K.
Solution 134.

Srinivasan, G.
Solutions 185, 186.

Subba Rao, K.
Solution 135.

Sriranganarayana, M. A.
Question 141.

Thebault, V.
Questions, 49², 51, 134, 184.
Solution 49², 135.

Venkatanarasimham, S.
Solutions 185².

Venkataraman.
Solution 185.

Venkatasubbiah, S.
Solution 184.

Weiland, A.
Solution 97.

Index of Questions Solved in Volume VIII

Question	Page	Question	Page
1597	96	1713	50
1668	97	1775	97
1694	49	1785	185
1700	134	1786	135
1708	49	1788	185
1791	185		

ERRATA FOR VOLUME VIII

Page	*Line*	*For*	*Read*
108	4B†	Brittannica	Britannica
114	19	$(n+2r)!$	$(2n+r)!$
128	2B	relation (4)	relation (4′)
128	8B	in (3′)	in (4′)
129	2	with (4)	with (4′)
129	11	by (4)	by (4″)
150	2B	ε	ε'
152	8	$1=np$	$=np$
156	14	$n^{\phi(m)}$	n_ϕ
156	15	$r \cdot n_1 n_2 \cdots n^{\phi}$	$r \cdot n_1 n_2 \cdots n_\phi$
157	10	after "(mod m')"	add "...(4)"
157	20	$p_s{}^t + \cdots)$	$p_s{}^t) + \cdots$
158	9	$\lambda m^{\phi_r} r$	$\lambda m \phi_r r$
159	8B	$\frac{1}{a_v} + \cdots$	$\frac{1}{a_v + \cdots}$
160		Delete the contents of line 3 and insert instead $\underset{*}{a_0} + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_r+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots\frac{1}{1+}\,\frac{1}{a_{r+n+1}} + \frac{1}{\underset{*}{a_p}}$	
160	7	Add at the end of the line "except unity"	
161	2B	(6)	(3)
162	2	(1)	(3·1)
165	17 and 19	$\xi_{p+2'+2}$	ξ_{p+2t+2}
166	6	ber	über
,,	22	No. 2	part X
,,	8B	Charles	Chasles
170		Turn figure so that X′OX is parallel to the shorter edges of the page	
175	8B	in (9)	in (8)
176	equation (14)	$E_2\,x$	$E_2\,x^2$
,,	9B	n^{th}	k^{th}
,,	2B	$E_2=1,\ E_4=5,\ E_6=61$	$E_2=\frac{1}{2!},\ E_4=\frac{5}{4!}\ E_6=\frac{61}{6!}$
,,	1B	$x^2+5x^4+61x^6$	$\frac{x^2}{2!}+\frac{5x^4}{4!}+\frac{61x^6}{6!}$

† Read "Fourth line from the bottom of the page.

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A PROBLEM ON RIGID MOTION

BY

F. W. LEVI, *Calcutta*

It is a fundamental fact of Analysis that every infinite set can be (1, 1)-represented by proper subsets of itself, whereas the same is impossible for finite sets.* If one considers rigid motions only in place of general (1, 1)-representations, an analogous proposition can be stated for linear point sets. Of course a bounded linear point set cannot be *transformed* by *any* rigid motion into a proper subset of itself, whereas certain unbounded sets can be as they contain proper subsets congruent to the original set. For example, the half rays $x>0$ and $x>1$ are congruent, and the rigid motion $x'=x+1$ transforms the first one into the second.

One may expect that the same proposition holds for point-sets of any dimension as the measure (area, volume etc.) of a bounded point-set is not altered by rigid motion, but in general point-sets have proper subsets of the same measure (the difference being not of positive measure) and it may be that these subsets are congruent to the original set.

The two-dimensional bounded point-sets which by a suitable rigid motion can be transformed into a proper subset, as well as these rigid motions themselves will be considered in this paper.

§ 1. *A characteristic example.*

Represent the points of the plane by complex numbers in the usual way, and consider the set

$$e^{2m\alpha\pi i} \qquad \dots \quad (1)$$

for $m=1, 2, \cdots$, α being any particular *irrational* number.

These points are all situated on the unit-circle, and different values of m correspond to different points. The rotation about 0 through the angle $2\alpha\pi$ transforms (1) into $e^{2(m+1)\alpha\pi i}$, which is a *proper* subset of it not containing $e^{2\alpha\pi i}$.

* This property has indeed been used by Dedekind (Was sind und was sollen die Zahlen, § 5) to define finite sets.

As this example is important for the general investigation given in this paper, it is useful to state the following property of the set (1).

The points (1) form a set which is dense everywhere on the unit-circle;

i.e. on every arc of the unit-circle there are points (1).

The factor $m\alpha$ in the exponent of (1) can obviously be replaced by

$$m\alpha - p \qquad \dots \quad (2)$$

where p is any integer.

The proposition therefore follows from the well-known fact that the numbers (2) form a set which for $m = 1, 2, \dots$ is dense everywhere in the interval (0, 1) —— and of course everywhere on the real axis. For the convenience of the reader, a proof of that statement will be given here.

Expand α as an infinite simple continued fraction, and let P_n/Q_n be the convergents; then for every odd n

$$0 < \alpha - \frac{P_n}{Q_n} < 1/Q_n^2$$

and therefore $$0 < \alpha Q_n - P_n < 1/Q_n.$$

As $\text{Lim } 1/Q_n = 0$, there exists for every $\varepsilon > 0$ such an n that $0 < \beta = \alpha Q_n - P_n < \varepsilon$.

Hence in every subinterval of (0, 1) of length $\geqslant \varepsilon$, there is at least one of the points $\beta, 2\beta, 3\beta, \dots$ and therefore a number (2). As this holds for every $\varepsilon > 0$, the set formed by the points (2) is dense everywhere in the interval (0, 1). Hence the set formed by the points (1) is dense everywhere on the unit-circle.

The corresponding statement holds for the set of the points

$$r\, e^{(2m\alpha + c)\pi i} \qquad \dots \quad (1')$$

as (1′) is obtained from (1) by a similarity transformation.

§ 2. *Transformations carrying M into a congruent proper subset.*

Let M be a bounded plane point-set which has congruent proper subsets. As it has been shown in § 1, sets of this kind exist.

By $\quad \mathbf{A}, \mathbf{B}, \dots \qquad \dots \quad (3)$

(with or without indices) will be denoted rigid motions of the plane carrying M into a proper subset; these form a system Σ.

$$\mathbf{A}(S) \qquad \dots \quad (4)$$

means the set in which a set S is carried by the motion **A**.

For every S, **A**(S) is congruent to S.

Especially

$$\mathbf{A}(M) \subset M, \ \mathbf{B}(M) \subset M.$$
$$\mathbf{B}(\mathbf{A}(M)) \subset \mathbf{B}(M) \subset M.$$

On the other hand **B** (**A**(M)) is congruent to **A** (M) and therefore to M.

If one therefore uses at first the transformation **A**, and then **B**, one gets again a transformation of the system Σ, which will be called **BA**. Thus

$$\mathbf{BA}(M) = \mathbf{B}(\mathbf{A}(M)) \subset M.$$

As usual, **AA** is replaced by $\mathbf{A}^2$ etc.

If a transformation **A** out of Σ is of the type of a reflection i.e. if its determinant is equal to -1, then $\mathbf{A}^2$, which also belongs to Σ is of determinant $+1$. There is therefore no loss of generality if one restricts the investigation to rigid motions in the *stricter sense*, i.e. to orthogonal transformations of the plane with determinant equal to $+1$. These motions are either translatory motions or rotations.

With a suitable system of co-ordinates, any translation is expressed by

$$x' = x + c, \quad y' = y.$$

As M is bounded, there exists an upper limit for the values of x, say x_0: this value is transformed into $x_0 + c$. For $c = 0$, the transformation becomes the identity; for $c \neq 0$, it implies the existence of points not belonging to M. Hence a parallel-displacement cannot carry M into a proper subset of M.

The transformations (3) must therefore consist of rotations. Let $2\alpha\pi$ be the angle of rotation corresponding to a rigid motion **A**.

If α is rational, say $\alpha = p/q > 0$ and p being an integer then $\mathbf{A}^q$ is a rotation through $2p\pi$; that is the identity, contrary to the supposition that **A** and therefore its positive powers carry M into a proper subset of M. Hence α is an irrational number. Let O be the centre of the rotation **A**, and $re^{c\pi i}$ be an arbitrary point of M, then

$$\mathbf{A}^n(re^{c\pi i}) = r\, e^{(2n\alpha + c)i\pi}$$

Hence every point (1′) belongs to M, and every point of the circle about O with radius r is a limiting point of M. By adding to M all its limiting points, one gets the *closure* [M] of M. From the

consideration cited above it follows that [M] consists of circles about O, and that therefore the smallest convex figure including M is a circle about O, if O is the centre of a rotation carrying M into a proper subset of it. Hence there cannot exist more than one such centre. *The rigid motions (in the strict sense carrying M into a proper subset of itself have all the same centre O; the closure [M] of M consists of entire circles about O.)*

Two rotations **A** and **B** about the same centre O through angles $2\pi\alpha$ and $2\pi\beta$ taken one after the other form a rotation through $2\pi(\alpha+\beta)$.

Hence

$$\mathbf{AB} = \mathbf{BA}.$$

The system Σ is composed of commutative rotations.

Consider any particular rotation **A** out of Σ, then there exists at least one point P which belongs to M but not to **A** (M). The inverse rotation $\mathbf{A}^{-1}$ carries P into a point not belonging to M; hence $\mathbf{A}^{-1}$ does not belong to Σ. The system Σ therefore does not form a group, but a *commutative half-group.** The elements of this half-group are rotations about a particular point O.

§ 3. *The half-group Σ and the group G.*

Consider the rotations

$$\mathbf{A}\,\mathbf{B}^{-1} \qquad \ldots (5)$$

where **A**, **B**. . . . belong to the half-group Σ.

As $$(\mathbf{A}\,\mathbf{B}^{-1})^{-1} = \mathbf{B}\,\mathbf{A}^{-1}$$

$$(\mathbf{A}_1\,\mathbf{B}_1^{-1})\,(\mathbf{A}_2\,\mathbf{B}_2^{-1}) = (\mathbf{A}_1\,\mathbf{A}_2)\,(\mathbf{B}_1\,\mathbf{B}_2)^{-1}$$

the rotations (5) form a (commutative group)† which will be denoted by G. As $\mathbf{A} = (\mathbf{A}^2)\,\mathbf{A}^{-1}$, every element of Σ is an element of G,

i.e. $$\Sigma \subset G.$$

Let P be a point of M. Consider the set M (P) of those points of M which are obtained from P by a suitable transformation out of G. It may be mentioned that not every element of G carries necessarily P into an element of M. Every element of Σ does, but there are elements of G not belonging to Σ – e.g. the identity and for

* A half-group is a system in which an associative composition of every pair of elements can be carried out.

† It is not necessary to check the associative law as the elements are transformations.

particular P, even every element of G – which have the same property. Let Q be a point of M (P) and R be a point of M (Q) then

$$Q = \mathbf{A}\,\mathbf{B}^{-1}(P), \quad R = \mathbf{A}_1\mathbf{B}_1^{-1}\,(Q);$$

hence

$$P = \mathbf{B}\,\mathbf{A}^{-1}(Q), \quad R = (\mathbf{A}\mathbf{A}_1)\,(\mathbf{B}\mathbf{B}_1)^{-1}(P).$$

The point-set M(P) is therefore uniquely determined by any one of its points

$$M(P) = M(Q). \qquad \dots \quad (6)$$

Furthermore $\mathbf{B}(Q) = \mathbf{A}(P)$ is a point of M. If on the other hand this equation holds for two points P and Q of M and two rotations **A** and **B** of Σ, then (6) holds. Hence two points of M belong to the same *class* M(P) if they can be transformed into the same point by suitable rotations of Σ.

If every element of G transforms P into a point of M, then the same holds for every point of M(P). In this case every element of G – and therefore every element of Σ – transforms M(P) into itself. Hence in this case no point of M(P) is *lost* by the rotations of the half-group Σ. If, on the other hand, one of the rotations (3) transforms M(P) into a set not containing P, then it transforms M(P) into a proper subset of M, and therefore into a proper subset of M(P). As M(P) = M(Q), every point Q of M(P) must be lost by a suitable rotation (3). The points M(P) are situated on the same circle about O. Certain considerations may be restricted to these constituent sets M(P), but it must be mentioned that the half-group of rotations transforming M(P) into a *proper* subset, may be a subset Σ′ of G which is different from Σ. Putting M(P) for M and Σ′ for Σ one gets again a subdivision of M(P) into classes and this procedure may not end after a finite number of steps.

§ 4. ***A set* M *corresponding to a given* Σ.**

The group G is a commutative group in which every element except the unit-element (the identity) is of infinite order. The power of this group is at most equal to the power of the continuum. Σ is a subset of G which forms such a half-group that every element of G is the quotient of two elements of Σ, and such that Σ does not contain the inverse of any of its elements. It will be proved that these conditions are sufficient too.

For this purpose, one has to apply the fundamental theorem of the general theory of point-sets that every set can be *well-ordered*. This theorem will be applied firstly to the set of all the real numbers, and then to any half-group Σ satisfying the above conditions.

Thus, let the real numbers be well-ordered in such a manner that the first element is the number 1.

$$\gamma_0 = 1, \gamma_1, \gamma_2, \ldots .$$

Subdivide this system into two sets A and B, where B contains those and only those elements γ_ν which are linearly dependent on a finite number of *preceding* elements with rational coefficients.

Hence 1 belongs to A, whereas every rational number $\neq 1$ belongs to B. Both the sets A and B are well-ordered sets.

That every element of B is linearly dependent on elements of A is seen in the following manner. If there were elements in B not dependent on elements of A, then there must be a first one say β_0. This element is dependent on preceding elements

$$\beta_0 = c_1\gamma_{\nu_1} + \ldots . + c_t\gamma_{\nu t}$$

Some of the elements γ_{ν_k} may be elements of A: those which belong to B are of a lower order than β_0 and therefore depending on elements of A. Hence β_0 depends on A; therefore every element of B depends on a finite number of elements of A. Every real number can therefore be represented in the form

$$c_0 + c_1\alpha_{\nu_1} + \ldots . + c_n\alpha_{\nu n} \quad \ldots (7)$$

where the coefficients are rational numbers, and $\alpha_{\nu\mathrm{I}}, \ldots, \alpha_{\nu n}$ are irrational elements of A. The representation (7) is unique; for if a real number has two representations of this kind, the difference

$$0 = d_0 + d_1\alpha_{\mu\mathrm{I}} + , \ldots + d_m\alpha_{\mu m}$$

must vanish. As $1, \alpha_{\mu\mathrm{I}}, \ldots, \alpha_{\mu m}$ are by definition linearly independent, the coefficients are all equal to zero.

Thus the elements of A form a *basis* of tho set of the real numbers. Every real number can be represented with this basis in one and only one manner in the form (7). The indices are not all integral numbers, but transfinite numbers; the basis forms a set of the power of the continuum.

Again let $\sum$ be a half-group being a subset of a commutative group G, and not containing the inverse element of any of its elements. The power of G is supposed to be at most equal to the power of the continuum, and its elements—except the unit-element

1—to be of infinite order. As 1 is its own inverse element, it cannot belong to Σ. Elements $\delta_1, \ldots, \delta_m$ are said to be *independent* if

$$\delta_1^{c_1} \cdots \delta_m^{c_m} = 1 \qquad \ldots (8)$$

implies $c_1 = \cdots = c_m = 0$. If however in (8) $c_m \neq 0$, then δ_m is *dependent* on $\delta_1, \ldots, \delta_{m-1}$. Let

$$\delta_1^{c_1} \cdots \delta_m^{c_m} = \delta^n = \varepsilon^n,$$

then $(\delta\varepsilon^{-1})^n = 1$. As an element of G which is of finite order must be equal to 1 it follows that δ and ε are equal. It is therefore not ambiguous to use fractional exponents* and to write

$$\delta = \delta_1^{d_1} \cdots \delta_m^{d_m}, \text{ where } d_k = c_k/n \qquad \ldots (9)$$

In multiplying two products with fractional exponents one has to add simply corresponding exponents. It must however be mentioned that not for every system of fractions $d_1, \ldots, d_m$, the formula (9) furnishes an element δ of G.

In exactly the same manner as the existence of a well-ordered basis of the set of the real numbers has been shown above, one can prove the existence of a *basis of* Σ, i.e. a well-ordered set of elements

$$\mathbf{A}_1, \mathbf{A}_2, \ldots \qquad \ldots (10)$$

by the help of which every element of Σ can be represented as a finite product

$$\mathbf{A}_1^{d_1} \cdots \mathbf{A}_s^{d_s} \qquad \ldots (11)$$

As (10) forms a subset of Σ, and therefore of G, its power is at most equal to the power of the continuum. Hence to every element (11) there corresponds an irrational number

$$\gamma = d_1 \alpha_{\nu_1} + d_2 \alpha_{\nu_2} + \cdots + d_t \alpha_{\nu_t} \qquad \ldots (12)$$

and furthermore a rotation about O, through the angle

$$2\pi\gamma \qquad \ldots (13)$$

As the irrational numbers α_{ν_k} are independent, to different elements (11) there correspond different numbers (12), and—as these numbers are irrational—different rotations (13). To the product of two elements

---

* See the author's pamphlet. Abelsche Gouppen mit abzählbaren Elementen (Leipzig 1917).—For recent investigations on groups of this kind, See R. Baer: Abelian groups without elements of finite order, Duke Mathematical Journal, 3 (1937).

of $\sum$ there corresponds the sum of the corresponding numbers (12) and therefore the rotation composed by the two corresponding rotations. The half-group $\sum$ is therefore *isomorphic* with the system of the rotations through the angles $2\gamma\pi$. Consider now the points

$$e^{2\pi\gamma' i} \qquad \dots \quad (14)$$

where γ' takes all the values (12) *and* the value zero. These form a set, say M, situated on the unit-circle. As the inverse of any element of $\sum$ does not belong to $\sum$, the negative of any number (12) is not a number (12) For $\gamma' \neq 0$, the point $e^{-2\pi\gamma' i}$ is therefore not a point of M. By every rotation (13) the set M is transformed into a congruent set, which is a subset of M as the sum of two numbers γ' is again a number γ'. This subset does not contain the point 1, as it would be the image of a point $e^{-2\pi\gamma i}$ which is outside M. Hence to every element of $\sum$ there corresponds a rotation carrying M into a proper subset of itself. On the other hand a rotation of this kind transforms 1 into a point (14), where $\gamma' \neq 0$ as the rotation cannot be the identity; hence it is a rotation (13). The given half-group $\sum$ is therefore isomorphic with the half-group of the rotations carrying M into a proper subset of itself. If P is an arbitrary point of M, then obviously

$$\text{M (P)} = \text{M}. \qquad \dots \quad (15)$$

From this formula it is evident that the type of M is in general not uniquely defined by M. It may be mentioned without proof that even if the half-group of rotations (13) is given there exist on the unit-circle, in general, essentially different sets M satisfying the condition (15).

§ 5. ***Measure of the lost points.***

Again, let M be transformed into a proper subset M′ by a rigid motion, and let L be the set of the "lost" points. If M is measurable then M′ and L are measurable and from

$$|\text{M}| = |\text{M}'| + |\text{L}|$$

and

$$|\text{M}| = |\text{M}'|$$

it follows that $|\text{L}| = 0$.

These equations hold equally whether the measure is taken in Jordan's or in Lebesgue's sense, and whether the measure is an areal measure or—in case M be situated on a single circle—an one-dimensional arc-measure.

If M is non-measurable the consideration just stated furnishes only a statement on inner measure:

$$\underline{M} \geqq \underline{M}' + \underline{L}$$

$$\underline{M} = \underline{M}'$$

Hence $$\underline{L} = 0.$$

Is it possible that the outer measure $\overline{L}$ is positive? Here one has to distinguish between the different definitions of measurement. In Jordan's sense the outer measure of a point-set is already positive when the set is dense everywhere in an interval. Sets M which lose subsets dense everywhere on the unit-circle can easily be constructed, e.g. the point-set $e^{2\pi\gamma i}$, where

$$\gamma = m\sqrt{2} + n\sqrt{3}, \qquad (m,n = 0,1,2,3 \cdots)$$

Indeed, by the rotation through the angle $2\pi\sqrt{2}$, this point-set is transformed into a subset thereby losing the points $e^{2\pi n\sqrt{3}i}$ for $n = 0,1,2 \cdots$, and these points are dense everywhere on the unit-circle. If we alter this example slightly by projecting the points on the circle from the centre, the lost points will be dense everywhere inside and on the unit-circle, and the outer areal Jordan-measure will be positive. The outer Lebesgue's measure however is zero. Thus the paradoxical result seems to be a consequence of the imperfection of Jordan's method of measurement.

It will however be proved now that there exist point-sets M which lose by certain rotations sets of positive outer Lebesgue's measure. Let $$1, \alpha_1, \alpha_2, \cdots$$
be a basis of the real numbers as in § 4. Then the angles of rotation are uniquely represented by $2\pi\gamma$, where

$$\gamma = c_0 + c_{\nu_1}\alpha_{\nu_1} + \cdots + c_{\nu_m}\alpha_{\nu_m}$$

the coefficients being rational, and $0 \leqq c_0 < 1$.

The points $e^{2\pi\gamma i}$ will now be distributed into an enumerable number of congruent subsets without common points

$$M_\iota, \quad \iota = 0, \pm 1 \pm 2, \ldots$$

which are defined by the inequality

$$\iota \leqq c_1 < \iota + 1.$$

obviously M_ι is transformed into $M_{\iota+n}$ by a rotation through $2\pi n\alpha_1$.

This shows the congruence of these sets. If they are measurable in Lebesgue's sense, they would be of equal measure, say $|M_0|$ and as a consequence of a fundamental theorem on Lebesgue's measure,

$$2\pi = |M_0| \sum_{-\infty}^{+\infty} n$$

which is impossible. Hence the sets are not measurable, and they have therefore a positive outer-measure.

Let now M be the set of those points $e^{2\pi\gamma i}$ for which $c_1 \geqslant 0$. By the rotation through the angle $2\pi\alpha_1$, the set M is transformed into a proper subset thereby losing the point set M_0 which has a positive outer Lebesgue's measure.

Ghoom (Darjeeling)
13-11-39.

GLEANINGS

Such a revolution in the rudiments of mathematical thinking does not go unchallenged. Brouwer's radical move to the left is speeded by an outraged roar from the reactionary right. "What Weyl and Brouwer are doing [Brouwer is the leader, Weyl his companion in revolt] is mainly following in the steps of Kronecker," according to Hilbert, the champion of the *status quo*. "They are trying to establish mathematics by jettisoning everything which does not suit them and setting up an embargo. The effect is to dismember our science and to run the risk of losing a large part of our most valuable possessions. Weyl and Brouwer condemn the general notions of irrational numbers, of functions—even of such functions as occur in the theory of numbers—Cantor's transfinite numbers, etc., the theorem that infinite set of positive integers has a least, and even the 'law of excluded middle,' as for example the assertion: Either there is only a finite number of primes or infinitely many. These are examples of (to them) forbidden theorems and modes of reasoning. I believe that impotent as Kronecker was to abolish irrational numbers (Weyl and Brouwer do permit us to retain a torso), no less impotent will their efforts prove today. No! Brouwer's program is not a revolution, but merely the repetition of a futile *coup de main* with old methods, but which was then undertaken with greater verve, yet failed utterly. Today the state [mathematics] is thoroughly armed and strengthened through labours of Frege, Dedekind, and Cantor. The efforts of Brouwer and Weyl are foredoomed to futility."

To which the other side replies by a shrug of the shoulders and goes ahead with its great and fundamentally new task of re-establishing mathematics (particularly the foundation of analysis) on a firmer basis than any laid down by the men of the past 2500 years from Pythagoras to Weierstrass.

E. T. BELL, *Men of Mathematics*.

GENERALISED GEOMETRY AND PHYSICAL THEORIES †

BY

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SUMMARY

(1) INTRODUCTION :—Relations between mathematics and physics—relation between geometry and physics—Relativity and geometrisation of physics—Quantum mechanics and abstract space—non-linear field theories.

(2) CLASSICAL MECHANICS :—Hamilton's characteristic function—configuration space—geometry of dynamics—principles of least action and least curvature—Phase space—dynamics and topology.

(3) RELATIVITY AND COSMOLOGY :—Unitary field theories—Stellmacher's work—The approximation method—Theory of supernovae.

(4) QUANTUM MECHANICS : -Hilbert space—Logic of quantum mechanics—Electron wave equation in general relativity and generalised spaces.

(5) ELECTROMAGNETIC FIELD THEORIES :—Quantum Electrodynamics—Born's field theory—Rosen's theory—Dirac's classical theory of the electron—Work of March and Foradori.

(6) NUCLEAR PHYSICS :—Mesotron theory—Born's reciprocity—Theory of Proca and Goudsmit—Wave geometry of the Japanese School.

1. Introduction

In the course of the development of modern theoretical Physics, the notions of generalised geometry have played a very prominent part. It is true that, of all branches of mathematics, Analysis has for a very long time been the principal weapon in the armoury of the theoretical physicist, and it is only recently that other branches are coming to be employed more frequently in the building up of the foundations of Physics and the erection of its superstructure. As examples in point one might mention the Birkhoff-Neumann logic of quantum mechanics (1), the algebra associated with it (2), the applications of group theory in wave mechanics and molecular physics (3, 4), considerations of probability and general statistics (5), and the weird programme of "*arithmetisation of Physics*" (6) initiated by Milne in his cosmological speculations.

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The connections between geometry and physics have, however, always been intimate. The idea that physical space is identical with three-dimensional euclidean space has dominated classical physics for a long time. With the advent of non-Euclidean geometry, an era of revolution was ushered in geometry which came to be placed on an axiomatic basis, and there appeared a temporary divorce between physics and geometry. The theory of relativity was responsible for again bringing about an intimate connection between them by stimulating investigations regarding the relationship of natural geometry, and abstract geometry. In fact, the golden age of generalised geometry—the subject to be dealt with in this symposium—might be truly said to have begun with the formulation of the general theory of relativity by Einstein, although it must be realised that this theory would probably not have seen the light of day had it not been for the creation of the absolute differential calculus by Ricci and Levi-Civita. For the first time, we have become familiar, through general relativity, with the fact that geometry, in so far as it introduces the determination of measure in the space-time world, has an essentially empirical character. This knowledge, culminating in the researches of Einstein on gravitational phenomena, and Weyl's attempt at the unification of gravitation and electro-magnetism, has given birth to the movement known as the "*geometrisation of Physics*" which aims at the construction of unitary theories reducing all physical phenomena of the field to a geometry of space-time or a space-time-something else, and has also been responsible for the opening up of a new field of geometry—the geometry of linear connection.

With the rise of Quantum Mechanics, the geometry of abstract spaces has reaped a rich harvest in Physics. The difficulties connected with the applications of quantum mechanics to field theories, with the positron theory, with the explanation of the mysterious specifically nuclear forces between nuclear particles, and with the mesotron theory have given rise to a number of geometrical theories of a Riemannian or non-Riemannian character, and even of the general topological type. Even the staid old classical dynamics is being dominated by these newer concepts of generalised geometry.

It might be that the physical ideas behind many of these theories are not of much significance in unravelling the mysteries of physics. But, they are nevertheless interesting in indicating methods of mathematical formulation which may probably be adopted in future successful theories. This is one of the reasons which have prompted the presentation of some of these theories in this address.

2. Classical Mechanics

The first significant geometrical idea introduced into classical mechanics was probably the most fruitful conception of Hamilton that the dynamical equations could be deduced on an optical analogy as the motion of a wave front from one position to a neighbouring position (7). This is really the basis of the transformation theory of classical dynamics according to which the canonical equations of motion simply express the fact that the whole course of a dynamical system can be regarded as the gradual self-unfolding of an infinitesimal contact transformation.

The most obvious geometrical method of expressing the motion of a dynamical system is to associate a "Configuration Space" M_n every point $(q_1, q_2 \ldots . q_n)$ of which denotes a state of the system, the motion being represented as a trajectory in this space. An alternative method is to think of a space of the (q_n, t) whose "world lines" denote the motion of the dynamical system. Using these ideas all the theorems of generalised dynamics can be expressed in a geometrical form.

Thus either in M_n or M_{n+1} the totality of world-lines or trajectories constitutes a group of $2n$-parametric curves. Through any point of M_{n+1} there passes a n-parametric bundle of curves only one of which passes through a neighbouring point, the bundle thus constituting a field. In M_n, however, there are ∞^1 trajectories through a point in a specified direction.

If the system be holonomic and without moving constraints with the kinetic energy in the form

$$T = \tfrac{1}{2}\, a_{ij}\, \dot{x}^i\, \dot{x}^j$$

we may assign to M_n the kinematical line-element (8)

$$ds^2 = a_{ij}\, dx^i\, dx^j$$

and thus obtain a Riemannian space R_n defined topologically and metrically by the dynamical system. The interest of this representation is twofold in that it not only gives a geometrical intertpretation to a dynamical problem but also gives a dynamical interpretation to multi-dimensional spaces or to spaces with a topology and metric not realisable in a Euclidean space. To consider two simple special examples, it might be mentioned that the dynamical system, consisting of a disc which can turn about an axis perpendicular to its plane with its centre at the same time free to move along a line in the plane, gives a "configuration space" which

is a flat space of two dimensions having the connectivity of a cylinder. Again, the system of two discs rotating independently gives rise to a flat space of two dimensions having the connectivity of an anchor ring (9).

Coming back to the general geometry of R_n, it is to be noticed that all vectors have covariant and contravariant components; in particular the covariant components of the velocity vector are the generalised momenta. Lagrange's equations of motion can be written in the form

$$\ddot{q}^r + \left\{ \begin{matrix} r \\ sv \end{matrix} \right\} \dot{q}^v \dot{q}^s = Q^r$$

considering the trajectories through a given point with specified directions $(dq_r/ds)_0$ those are determined by the initial velocity of projection $v_0 = (\dot{s})_0$, the variation of which alters merely the *geodesic curvature* of the trajectories. This curvature K_g can be obtained in the form

$$K^2_g = \frac{1}{\dot{s}^4} \left\{ g_{sr} Q^s Q^r - \left(Q_r \frac{dq_r}{ds} \right)^2 \right\}$$

and the equations of motion reduce to

$$\frac{d^2 q^r}{ds^2} + \left\{ \begin{matrix} r \\ pv \end{matrix} \right\} \frac{dq^p}{ds} \frac{dq^v}{ds} = \frac{1}{\dot{s}^2} \left\{ Q^r - Q_p \frac{dq^p}{ds} \frac{dq^r}{ds} \right\}$$

which shows that the geodesic lines (with $K_g = 0$) are the limiting positions of the trajectories for $\dot{s} = \infty$. (10). In the case where the external forces are absent, K_g is identically zero and the trajectories are the geodesic lines of the Riemannian space. It might also be mentioned that the K_g introduced here is the same as the curvature of the Gauss-Hertz law.

For systems with constant energy we might consider, instead of the kinematic line-element, the *action line-element*

$$ds^2 = 2\ (h - v)\ T dt^2 = g_{ij}\ dq^i\ dq^j$$

and the principle of least action is equivalent to stating that the curves of natural motion are the geodesics in this Riemannian space. Hamilton's variation principle can be shown to be the same as this principle translated to the space-time (M_{n+1}).

Besides the notion of configuration space, it is also possible to associate with a dynamical system a "*phase-space*" $\sum$ of $2n$ - dimensions every point of which corresponds to the choice of n co-ordinates of position and the n conjugate momenta. Equally useful is the setting up of the $(2n+1)$ dimensional phase-space $\sum'$ including t as well. In either case the equations of motion can be considered as

laws of propagation inducing a steady fluid motion in the phase-space. The existence of the integral invariant of order $2n$.

$$\int\int \cdots \int \delta p_1 \cdots \delta p_n \, \delta q_1 \cdots \delta q_n = \text{const}$$

for the canonical system of equations is equivalent to the statement of conservation of volume in this fluid motion. With this representation it is possible to give a geometrical meaning to the Hamilton-Jacobi theory, as well as to the dynamical theorems based on Lie's transformation group theory.

The most fruitful results arising as consequences of the notion of the phase-space are perhaps the investigations relating to dynamical motions "im Grossen". Such questions raised for the first time by Poincaré relate to the totality of all possible forms of motions and the complete course of any such particular form. Since the manifold of configurations of a dynamical system correspond to a set of points in phase-space, the above type of questions requires topological methods for its solution. These have been developed by Birkhoff (11) who has considered the simplest types of motion chosen in as simple a way as possible. With the growth of topology in recent years there has also been consequent development in this branch of the subject specially with regard to a type of motions known as *transitive*. A dynamical system is *regionally transitive* if there exists a motion such that the points of phase-space defined by this motion constitute a set which is everywhere dense in the phase-space, and such a motion is called *transitive*. If the set of points be measurable and either of measure zero or measure equal to that of the phase-space, the system is said to be *metrically transitive*. As a typical theorem in this field might be mentioned a result due to Hedlund (12) that the systems of geodesics on a certain set of surfaces which are closed, orientable, of arbitrary genus greater than one, and of constant negative curvature are metrically transitive.

3. Relativity and Cosmology

The general theory of relativity is one of the landmarks in the history of the relations between geometry and physics. It is not proposed to deal here with the classical aspects of either the special or general theories of relativity, but it would be worthwhile to say a few words regarding recent work in the domain of unitary field theories seeking to unify gravitational and electromagnetic phenomena into a single geometrical scheme. Of the several attempts so far made in this direction two of the simplest and most natural ones are those due to Weyl (13) and Kaluza and Klein (14) using

respectively the notions of gauge-invariance, and a fifth dimension. Even the celebrated attempt of Einstein-Mayer (15) introducing the notion of "teleparallelism" must be considered rather artificial. Recently, however, Einstein and Bergmann (16) have proposed a slight modification of the Kaluza theory which might be noticed here.

The Kaluza-Klein theory introduces a fifth dimension in attempting to derive a unitary theory connecting gravitation and electricity. Einstein has attempted to generalise this theory by putting in physical concepts into the purely mathematical structure of Kaluza's theory.

The aim of this theory of Kaluza was to obtain some new physical aspect for gravitation and electricity by introducing a unitary field structure with the aid of a fifth dimension, the essential result being that such a five-dimensional structure could be built up so as to be equivalent to a four-dimensional structure *plus* a vector field which is the potential vector for the electro-magnetic field. This result, though elegant mathematically, was not productive of new physical ideas, and consequently many attempts were made to retain the essential formal results obtained by Kaluza without sacrificing the four-dimensional character of the physical space. But all such attempts have proved unproductive, and it appears impossible to formulate Kaluza's idea in a simple way without introducing the fifth dimension.

On the basis of these considerations, Einstein and Bergmann have now attempted to introduce the fifth dimension in a very effective manner without its being merely a sort of "catalytic agent" as in the Kaluza theory. To bring out clearly the generalisation proposed by Einstein, let us consider how Kaluza's five-dimensional structure is made equivalent to a four-dimensional one and a vector field. It can be shown that by a suitable characterisation of the 5-space with the metric,

$$d\sigma^2 = \gamma_{\mu\nu} dx^\mu dx^\nu \qquad (\mu, \nu = 0, 1, 2, 3, 4)$$

the components of the fundamental metric tensor can, by the choice of a special co-ordinate system, be reduced to ten functions g_{mn} and the four functions A_m (m, $n = 1, 2, 3, 4$) which do not depend on x^0. This reduction gives a four-dimensional description of the space, and the independence of the functions on x^0 shows the purely formal nature of the fifth dimension x^0 which just put in only to be taken out later. On Einstein's new theory it is shown that, with a suitable

modification of the postulates of the 5-space, it is possible to make an exactly analogous reduction to g_{mn} and A_m with this difference that the components of g_{mn} are in general periodic functions of x^0. The A_m, however, is independent of x^0 as in the old theory. Remembering that g_{mn} is a four-dimensional metric tensor, this amounts to an intimate physical connection of the space-time with the new dimension. The x^0 which is put in at first is not taken out after the reduction but left behind so as to modify the 4-metric. The periodicity of the components of this 4-tensor in the new co-ordinate enables one to interpret physically the fifth dimension. In a very rough way, one could describe this as a sort of a phase, giving rise to the 5-dimensional space-time-phase. Since, however, this new co-ordinate is "dimensionless" there arises no contradiction with the empirical four-dimensional character of physical space.

From its very nature, the new theory is essentially complex in its physical aspects, and Einstein and Bergmann have given the derivation of the fourteen field equations starting from a variational principle, and also the indentities satisfied by the field equations. The theory involves four universal constants of which one corresponds to the gravitational constant involving a connection between the units of length and mass, another depending on the unit of length, while the remaining two are "genuine" universal constants which cannot be eliminated from the theory.

When looked at from the purely geometrical point of view the new theory introduces some very interesting features. The five-dimensional space defined by the metric is here closed with respect to one dimension, and this closed space will be represented by a space which is open and periodic with respect to this dimension. A point P of the physical space will be represented by an infinite number of points P. P′, P″ of the 5-space. This type of non-homeomorphic correspondence between general metric spaces is itself a rich mathematical concept capable of a large number of developments.

Apart from the above geometrical unitary theories, one might consider a more fundamental unitary tendency in attempts at unifying the notions of field and matter. All the field laws of physics, gravitational, electromagnetic and quantum, are structure laws which break down for very great concentrations of energy or where sources of the field are present. It has been obvious from the evolution of the history of physics that it cannot be built on the matter concept alone. The alternative to this is to find out if we could not reject the matter concept entirely, and build a pure field physics which

explains nature by structure laws valid always and everywhere. This programme, has, so far, not been fulfilled in spite of the great advances made in quantum mechanics in recent times. With a few such unitary theories on the electromagnetic side we shall have occasion to deal later (§ 5). For the present, so far as it concerns gravitation, an important question that arises is whether the motion of matter considered as point-singularities of the field could be determined by the gravitational equations alone. This question has been answered in the affirmative by Einstein, Infeld and Hoffmann (17) who have set up an "approximation method" in the treatment of the problem, and have also applied it successfully to the relativistic problem of two bodies.

A geometrical treatment of the motion of singularities has also been attempted by Stellmacher (18) who has systematically applied Hadamard's method of characteristics and bicharacteristics to the field equations of gravitation as well as the Maxwell-Lorentz equations in the field of general relativity. Also the law according to which the singularities are propagated has been derived and this leads to the physically significant result that in the theory of characteristics it is essential to speak both of gravitational and electrical quanta. A treatment of gravitation alone appears meaningless from this standpoint. There exists however a remarkable relation between gravitational and electrical energy for which specially great stability results.

Coming to cosmological theories, it would perhaps be not quite wrong to describe them as relativistic theories of idealised geometrical models. Static and non-static models with specialised line elements and pressure-density distributions have been constructed and studied in great detail, and applied to the explanation of the fundamental phenomenon of nebular recession (19). The remarkable theory of Milne (20), which too had a cosmological origin, is in reality an alternative to the programme of geometrisation of physics by the substitution of kinematics for geometry.

As an example of a cosmological theory, which dispenses with the current idealised models and breaks new ground, might be mentioned a theory proposed by Zwicky (21) to explain the tremendous generation of energy in the recently discovered supernovae. Suggesting that the nova outburst is the result of the transformation of an ordinary star into a "*Collapsed neutron star*" with density of the order of 10^{14} gm/cm^3, i.e. of the order of density inside atomic nuclei, he develops the purely geometrical idea that the *very properties of space in highly collapsed stars may be radically different*

from the properties of ordinary space. The classical Schwarzschild theory on which the usual stellar models are based even if combined with the fundamental result, $E = mc^2$ of special relativity is unsatisfactory in this case, in that it does not allow us to formulate clearly how the mass equivalent of the gravitational potential energy is to be distributed over space. This difficulty has its real root in the fact that gravitation is a *co-operative phenomenon,* and the general theory of relativity alone must be used to estimate such effects as the effective masses of large dense stars.

This theory of supernovae gives rise to another cosmological aspect of the theory of neutron stars viz., the conclusion of Stern (22) on the basis of statistical mechanics and thermodynamics that in a state of equilibrium between matter and radiation in the Universe, practically all matter would be found to have been transformed into radiation. The ratio of concentration of matter transformed into radiation to the concentration of matter in the ordinary forms is determined on Stern's Theory by the undefeatable magnitude $e^{6\times 10^7}$ or, in other words, with radius of Universe R equal to 6×10^{39} cm., and number of electrons $1{\cdot}2\times 10^{78}$, not even a single electron could exist in R in statistical competition with radiation. As an escape from this, Zwicky has suggested consideration of matter, not in the ordinary forms, but in the clustering form as in neutron stars where the velocity becomes zero, and the gravitational packing energy of a particle of mass m may become equal to its original proper energy.

If the neutron-star hypothesis of the origin of super-novae can be proved, it will be possible to subject the general theory of relativity to a really crucial test. This theory, although profound and extremely satisfactory in its epistemological aspects, has so far practically not lent itself to any very obvious and generally impressive applications. It may thus be possible to remove the unfortunate discrepancy between the formal beauty of the general theory of relativity, and the meagreness of its practical applications.

4. Quantum mechanics

In geometry there have been two essentially distinct tendencies the abstract or axiomatic, and the constructive or analytical. Of these it might be said that the former has played the predominant role in enriching the conceptions of geometry. Thus the transition from Euclidean to Non-Euclidean geometry has been due to an abstract idea. Similarly the geometries of Riemann and Cartan, on

the former of which Relativity has been based, are the results of generalising the abstract notions of curvature and torsion. In the development of the geometry of the infinite dimensional complex space which is the geometry appropriate to quantum mechanics, the suggestions of generalisation have, however, come from the purely analytical side, in fact have been based on algebraic analysis viz., the question of linear orthogonal transformations of a quadratic form due in its essence to Hermite. But the geometry itself can be developed in a purely axiomatic way (23). We can first construct axiomatically the ∞-dimensional real non-complex space starting with a simple generalisation of Euclidean space of n dimensions. The axioms of affine geometry, the idea of linear operators could first be introduced, and then the metric notion by the definition of the scalar product of two vectors, and the notion of orthogonality. The transition to the case $n \to \infty$ offers no difficulty from the abstract point of view, the associated difficulties arising only in an analytical treatment. The generalisation to the complex space follows on exactly parallel lines, and all the above notions can be retained intact. The last step is the reduction of the abstract geometry so obtained to the analytical form by the use of co-ordinates based on the notion of Hermitian symmetry thus showing that the complex metric geometry obtained is essentially Hermitian with its operators also Hermitian.

To show that this is the appropriate geometry for quantum mechanics it is necessary to mention that quantum mechanics shares along with classical mechanics, and classical electrodynamics the notion of phase space already mentioned (§ 2). According to this concept any physical system S is at each instant hypothetically associated with a "point p" in a fixed phase-space $\sum$ and this point is supposed to represent mathematically the "state" of S ascertainable by "maximal" observations [23, § 4 (1930)]. Furthermore the point p_0 associated with S at a time t_0 together with a prescribed "mathematical law of propagation" fix the point p_t associated with S at any later time t.

Thus in classical mechanics $\sum$ is a region of ordinary $2n$-dimensional space. In electrodynamics the points of $\sum$ can only be specified after certain functions such as the electro-magnetic and electro-static potentials are known, and hence $\sum$ is a *function space* of infinitely many dimensions. Similarly in quantum mechanics the points of $\sum$ correspond to "wave functions" which are complex and functions of space and time and hence $\sum$ is again a *function-space* and a complex space of infinitely many dimensions. The law of

propagation is given by Schrodinger's equations just as Maxwell's equations give the law for electrodynamics. Using the analogy of fluid motion induced by the law of propagation, we might say that while the flow conserves volume in classical mechanics (§ 2), it conserves distances in quantum mechanics since the equations are unitary.

A difference might immediately be noticed between the physical quantities that represent a point in phase space in quantum mechanics and electrodynamics. While in the latter we might say that the laws of propagation i.e. Maxwell's equations determine the electromagnetic field, the interpretation in the former case is of a more abstract nature. The equations of quantum physics, while they are structure laws just like Maxwell's and Einstein's equations do not determine concrete things like the electromagnetic or gravitational fields but the abstract *probability waves*. Thus even in the case of the one particle problem, while we can speak in classical mechanics of the positions and velocities of the particle we can only consider, in quantum mechanics, probability waves in a three-dimensional continuum. For n particles the probability waves will be functions in a continuum of $3n$ dimensions, but the phase space is again a function space of infinity of dimensions.

Further analysis is however necessary to specify more exactly the nature of this function space so as to be in consonance with the principles of quantum theory. A penetrating analysis of the question has been given by J. v. Neumann (24) who has shown that, as a consequence of the equivalence of the matrix and wave mechanics points of view, this function space is isomorphic with a *Hilbert space* R_∞. The essence of the matrix point of view is the reduction of the Hamiltonian H to the diagonal form $S^{-1}HS$ which leads to the "eigenwerte" problem

$$\sum_{\nu} h_{\mu\nu}\, x_\nu = \lambda x_\mu$$

while the corresponding "eigenwerte" problem of wave mechanics is

$$H\psi(q_1 \cdots q_n) = \lambda \cdot \psi\,(q_1 \cdots q_n)$$

H being considered a differential operator. Thus we have two typical spaces, the discrete space Z with index values (Z = 1, 2, . .) and the continuous configuration space Ω of the physical system. The equivalence of the two theories does not mean the equivalence of Z and Ω, but that of the sequences $(x_1, x_2, \ldots)$ which are functions in Z and the wave function $\psi\,(q_1, q_2 \ldots q_n)$ which are functions in Ω i.e. of the function spaces F_Z and F_Ω. Since the

normalising condition $\sum_{\nu} |x_\nu|^2 = 1$ is assumed in F_z this function space is the usual *Hilbert space.* That F_z and F_Ω are isomorphs is merely an alternative way of expressing the well-known *Riesz-Fischer* theorem.

It has been noticed above that the ∞-dimensional complex space is the appropriate one for quantum mechanics, but the dimensionality axiom has not been rigorously examined. We might consider an abstract space R satisfying the following axioms:

A. R is a linear space.

B. In R a Hermitian scalar product is defined.

$C.^{(n)}$ There are exactly n linearly independent vectors in R. Then R would be the n-dimensional complex Euclidean space. If however the axiom $C^{(n)}$ be replaced by $C^{(\infty)}$ i.e. that there exists no such maximum number as n, then this really does not amount to an axiom at all, and hence in order that our space R should determine the Hilbert space R_∞ it is necessary to impose two other axioms D and E viz.,

D. R is a complete space.

E. R is a separable space.

For R satisfying A, B and $C^{(n)}$, the axioms D, E follow as consequences, but for R to define R_∞, D and E are essential.

Having axiomatically defined the Hilbert space R_∞ (F_z) its geometry could be developed analytically, and the notion of operators introduced. Lastly the introduction of the statistical postulate of probability amplitudes completes the geometrical scheme of quantum mechanics, and the correspondence between the geometrical magnitudes and their physical interpretation.

It will be evident from the above considerations that the foundations of quantum mechanics are essentially of a geometrical character, and this conclusion has been further strengthened by the work of Birkhoff and J. v. Neumann (2) who have examined the logic of quantum mechanics, and built up a calculus of propositions for quantum mechanics on the basis of the facts that all operators are Hermitian, and all Hermitian-symmetric operators in Hilbert space correspond to observables. A careful examination of this calculus has led to the conclusion that it has the same structure as an *abstract projective geometry,* while in classical mechanics the experimental propositions constitute a *Boolean algebra.*

Amongst one of the unsolved problems of quantum mechanics is its fusion with the theory of relativity. In all arguments about the transition from classical to quantum physics, only the pre-relativistic description is used in which space and time are treated differently. If, however, we try to begin from the classical description as proposed by the relativity theory then our ascent to the quantum problem is much more complicated. A partial solution of this problem is contained in the relativistic wave equation of the electron constructed by Dirac. This has enriched the concepts of geometry by showing that Lorentz invariance and tensor invariance are not necessarily identical, the former being more suitably expressed in terms of Spinors which play such an important part in the geometrical theory of projective relativity. The Dirac wave equation has also been the starting point of a series of researches of the highest importance to generalised geometry. A general invariant representation of Dirac's wave equation of the one-electron problem on the basis of Einstein's general relativity was first given by V. Fock (25) who investigated the wave-mechanical behaviour of the motion of an electron in an arbitrarily specified electro-magnetic and gravitational field. This theory has been further developed by Schrodinger (26), Schouten (27), Bargmann (28), Infeld and Van der Waerden (29). The representation of a corresponding wave equation for Einstein-Mayer's unitary field theory has been investigated by Pauli and Solomon (30), and, in a different mathematical way by Schouten and Van Dantzig (31) and also by Pauli (32). In the same category might be placed the work of Dirac (33) who has attempted to set up electron wave equations in de-Sitter space and conformal space The geometrical interest in the extension to various spaces lies in the fact that the equations of atomic physics have to be so formulated that they form a scheme which *is invariant for groups of transformations* which carry over the space into itself. Nearly all the more general spaces have only trivial groups of operations of this nature, but the above two spaces are exceptions. Defining de-Sitter space as the surface of a four-dimensional "sphere" (of a hyperbolic character in one dimension) imbedded in five-dimensional space, Dirac has shown that all the principal equations can be expressed in tensor form involving indices taking on five values. For conformal space defined as the four dimensional surface of a hyper-quadric in five dimensional projective space, it is found, on the other hand, that the tensor-equations require the use of six valued indices, and, moreover, not all equations of physics but only those that are invariant under conformal transformations of the ordinary space-time can be

generalised to conformal space. In particular it is found that the electron wave equation cannot be so expressed while it is possible for Maxwell's equations giving an indirect proof of the result due to Cunningham and Bateman (34) that Maxwell's equations are invariant under conformation transformations.

From the physical point of view, however, the above generalised theories introduce additional terms in Dirac's equation the deviations of which from the ordinary equation are so small as to be hardly within the range of experimental determination.

5. Electro-magnetic field theories

We have mentioned above one of the fundamental difficulties besetting quantum theory viz., the failure to bring about a general and natural fusion between relativity and quantum theory. Another equally important difficulty is the question of the interaction between elementary quanta of matter and the electro-magnetic field, or the so-called quantum electrodynamics. In the many particle problem, we have seen that the description of quantum phenomena could be achieved by considering probability waves in space of $3n$ dimensions. But for the quantum description of the field it would be necessary to consider a probability wave with an infinite number of dimensions, and this transition from the classical field concept to the quantum concept is a very difficult step, and all the theories so far developed in this direction are to be regarded as unsatisfactory.

Let us consider in a little more detail one such theory which is provisionally widely accepted viz., the Heisenberg-Pauli Theory (35) which includes as a particular case all other previous theories including those of Dirac and Fermi (36). This theory has been shown to be eminently successful within well-defined limits (37) in explaining phenomena of the quantum theory of radiation. In essence, this theory is a generalisation of Dirac's theory of radiation in what it considers a quantum theory of the total field including the electro-static. In the division of the field into plane waves (i.e. spatial Fourier analysis) into transversal and longitudinal waves, it can be shown that the first determine the characteristic light waves and the second the static interactions the latter replacing the longitudinal waves which do not lead to any new results on quantisation.

The difficulties of this quantum-electrodynamics appear to be of two sorts. The first can be indicated by considering, for the moment, the force-free particles of the Dirac's wave equation, and by coupling of these electrons with the electromagnetic field we immediately have

difficulties connected with negative energy states. Even for a quantum electrodynamics *in vacuo* the results are not valid for wave lengths $\lambda < \lambda_0$ $(= h/mc)$, the Compton wave-length; and if valid for $\lambda < h/mc$, the field of interaction must be very weak. The other difficulty is that the theory leads to an infinite self-energy of the electron, and wave lengths $\lambda < r_0$ $(= e^2/mc^2)$ are excluded from the theory. The second difficulty is perhaps not exactly quantum-theoretic in its origin; in fact, the classical (Lorentz) theory of the electron yields, when the electron radius goes to zero, an infinite quantity, the 'electro-magnetic mass". In other words, a point electron cannot, on account of the infinite inertia of its associated electro-magnetic field, be accelerated by a finite external force. In the Heisenberg-Pauli theory also which starts with point electrons one obtains for the electrons an infinite inertial mass (38).

To get over this second difficulty several theories have been proposed. One such theory is Born's non-linear field theory (39) which, although it has not been able to achieve the purpose for which it was intended, has nevertheless added new ideas to quantum electrodynamics and given a beautiful solution to the old classical problem of the structure and inertial mass of the electron. This theory is unitary in its nature in that it considers an electron as a point singularity in an electrodynamic field, and Maxwell's equations of the field are modified so as to make the energy of the field around the singularity representing the electron's self-energy a finite quantity. It has been known for quite a long time that in the self-energy problem the infinity could be got over by the assumption of an extended electron, but then the theory would become non-invariant relativistically. For, suppose we introduce a form function $D(x, y, z)$ to give the spatial extension of an electron such that the function is different from zero only in the inside of a small region; then by changing into another co-ordinate system the time t appears in this function, and it becomes impossible to formulate the field laws invariantly. In addition to this difficulty, the assumption of an extended electron would introduce a good deal of arbitrariness (in the choice of charge distribution) in the theory and would also go against the postulate of atomistic constitution of electricity. Born's field theory overcomes all these objections since the equations of the theory are relativistically invariant. It might also be noticed that Born's field theory is also gauge-invariant which condition has been shown to be essential by Weyl for any satisfactory field theory. This is achieved in Born's field theory since the Lagrangian used in

the variational principle on which the whole theory is based contains only the field strengths and not the potentials.

Considering first the classical part of the theory, one starts with the Lagrangian $L(g_{kl}, f_{kl}) = L(F, G)\sqrt{-g}$, where g_{kl} and f_{kl} are the metric and electro-magnetic field tensors, and F, G the usual invariants $\frac{1}{2} f_{kl} f^{kl}$ and $\frac{1}{4} f_{kl} f^{*kl}$. As usual the existence of the potential ϕ_k is assumed so that

$$f_{kl} = \frac{\partial \phi_l}{\partial x^k} - \frac{\partial \phi_k}{\partial x^l}$$

giving the identity

$$\frac{\partial \sqrt{-g}\, f^{*kl}}{\partial x^l} = 0 \qquad \text{(i)}$$

We introduce a second kind of antisymmetrical field tensor p_{kl} by the relation $\sqrt{-g}\, p^{kl} = \partial L / \partial f_{kl}$, and the variation principle for the Lagrangian gives the Eulerian *equations*.

$$\frac{\partial \sqrt{-g}\, p^{kl}}{\partial x^l} = 0 \qquad \text{(ii)}$$

(i) and (ii) are the complete set of field equations. The energy-impulse tensor is given by

$$\sqrt{-g} \cdot T_{kl} = -2\partial L / \partial g^{kl} = \sqrt{-g}\,(L g_{kl} - f_{ks}\, p_{lr}\, g^{sr}).$$

It is easy to generalise the action principle in such a way that it contains Einstein's gravitational laws; one has only to add to the action integral the term $\int R\sqrt{-g}\, d\tau$, where R is the scalar of curvature.

Leaving out of account the gravitational laws, the essence of the theory consists in not taking the charge distribution ρ as fundamental but two sets of field strengths **E**, **H** and **D**, **B**, the equations connecting these involving a certain universal constant b (limiting field strength) connected with the classical electronic radius r_0 by $b = e/r_0^2$. For weak fields $\mathbf{E} \equiv \mathbf{D}$, $\mathbf{H} \equiv \mathbf{B}$. By *imposing the* condition of self-conjugacy on the action function used in the variation principle, Born has shown that these four equal representations of the theory according as (**B**, **E**), (**D**, **H**), (**B**, **D**), or (**E**, **H**) are taken as fundamental. With the third or the **U**-representation in the special form (Born's action function)

$$L = \sqrt{1 + \frac{F}{b^2} - \frac{G^2}{b^4}} - 1$$

the energy-density u is given by

$$4\pi u = b^2\{1 + (\mathbf{D}^2 + \mathbf{B}^2)/b^2 - [\mathbf{DB}]^2/b^4\}^{1/2} - b^2$$

and the total energy and momentum **U** and **G** here form a four-vector, thus successfully exploiting the notion of electro-magnetic origin of mass.

The field equations applied to the special case of an electro-static field with spherical symmetry, and a singularity at the origin, give for the self-energy of the electron the finite value mc^2. Another important case where the field equations have been solved is that of a ring-singularity (40) where also one obtains finite values for the total energy and angular momentum.

In the further development of Born's field theory, the question of the form of the action function has been discussed in great detail. A great number of action functions have been proposed and they all appear to serve the purpose of giving finite energy for the electrical particle equally well. Even the symmetry conditions of self-duality and the conditions of existence of simple algebraic relations between the f_{kl} and p_{kl} – fields do not uniquely specify the action functions. It has been shown in fact (41) that there exist a two-fold infinity of such action-functions.

Coming now to the quantization of the field equations, Born has attempted the same on the lines of the Heisenberg-Pauli theory, taking **D** and **B** as the fundamental variables in place of the potentials ϕ_k. A coherent theory has been built up but its successes must be considered very meagre indeed. The most formidable objection to the quantized theory is the result of Pyce (42) that the electromagnetic angular momentum has its eigen-values as integral numbers thus showing that the theory cannot explain spin. Another purely classical attempt to explain spin of the electron on this theory has met with the same fate (43).

A remarkable coincidence between Born's field theory in its classical aspects, and the purely quantum theory of the positron is worth noticing In the later theory, for the region $\lambda < \lambda_0$ the field equations can no longer be taken to be linear, and remarkably enough all calculations made hitherto confirm the fact that variations from linearity can be represented by a universal Lagrangian of the same form as that assumed in the Born field theory. The deviation from the superposition-principle is unimportant as long as the field strength remains less than a certain critical value which is remarkably almost equal numerically to the field strength e^2/r_0^2 at the boundary of the electron of radius r_0 on the classical Lorentz-theory. This must have a deep significance, but no satisfactory explanation has so far been advanced.

The most serious objection that can be brought against a theory which postulates an electromagnetic origin of mass is the existence

of the neutron, and also the theory of the positron in which negative and positive mass play symmetrical roles. This is not consistent with a theory which does not envisage negative mass even abstractly. Another serious drawback in the non-linear field theory is the absence of a definite principle which serves to specify the action-function to be used in the variation principle.

Rosen (44) has attempted a non-linear field theory by attempting to get rid of singularities in the field. For such a theory he has concluded that the action function L should also contain the potentials ϕ_k besides the f_{kl}, but in order that a gauge-transformation

$$\phi'_\mu = \phi_\mu + \partial\lambda / \partial x_\mu$$

may not change the form of L or the field equations, he has introduced other arbitrary parameters so as to conserve gauge-invariance. By a suitable choice of these arbitrary terms in the form of a scalar function ψ such that it is transformed according to the law

$$\psi' = \psi e^{i\varepsilon\lambda}$$

a consistent theory can be built up in which the non-linear field equations resemble the Gordon-Klein relativistic scalar wave equation. Application to the static spherically symmetric case leads however to a negative mass of the electron.

Another purely classical theory has been attempted recently by Dirac (45) which is not based on any kind of model, retains the Maxwell's equations as they are but tries to remove the infinities by a subtraction formalism of the type used in position theory giving the finite mass as the difference between the infinite negative mass at the centre and the infinite positive mass of the surrounding Coulomb-field. The motion of an electron as disturbed by an electro-magnetic impulse is worked out on the basis of the theory and leads to results, of great significance viz., that it is possible for a signal to be transmitted faster than light through the interior of an electron. The finite size of the electron reappears here in a new sense, the *interior of the electron being a region of failure*, not of the field equations of electro-magnetic theory, but of some of the elementary properties of space-time, although the whole theory is Lorentz invariant. It has been remarked (46) that this theory is not really a solution but an evasion of the problem of explaining the elementary masses since it merely alters the definitions of force, momentum and energy as given in Lorentz's classical theory of the electron. Nevertheless, it might be pointed out that geometrically we have here a very intriguing situation in as much as the interior of the

electron is now a region where the geometry and not the field laws, appears to break down.

Related to the problem of self-energy, and the question of space-time in the interior of an electron is a theory developed by A. March (47) *in a series of papers on the geometry of a "Kleinster Raum."* It is pointed out that quantum mechanics only makes a sparse use of the notion of metric in geometry in as much as it avoids electrons describing definite orbits. Where, however, it actually uses the metric, as for example in the self-energy of an electron or the gravitational energy of a photon, it immediately resolves itself into contradictions. The fundamental notion of March's theory is that the length of the curve-element between two points is to be replaced by

$$\int \left\{ \sqrt{g_{ik} dx^i dx^k} - \gamma \right\}$$

this length being taken in the statistical sense as the result of very many measurements, γ being a small universal positive scalar invariant of the co-ordinates. Thus a statistical metric is introduced which does not permit the use of lengths less than a certain limit γ. The essential nature of the statistical condition is brought out from the fact that, without it, the extension proposed would be merely trivial. Thus if P_1 and P_2 be two points such that the length $P_1P_2 < \gamma$, we can consider a third point P_3 where P_1P_3 and P_2P_3 are $> \gamma$ and define $P_1P_2 = P_1P_3 - P_2P_3$, but a statistical definition would remove this contradiction by showing that P_1 and P_2 cannot be distinguished from each other.

From the mathematical point of view, this geometry would be of quite a complicated type. Thus if $P_1, P_2, \cdots\cdots P_n$ be a series of points on a line at distances γ, then $P_1P_n = 0$. This contradicts the axioms of Riemannian geometry, and it means we are here dealing with a non-Riemannian geometry where the metric is defined in the integral and not the differential form. In this geometry we might have, for example, *circles of zero-radius but a finite circumference*. In order to avoid these unnecessary complications, March introduces the notion of "physical points" in place of the mathematical point and like it also extensionless, but with the addition of the statistical concept. Thus all the formal representation of structural natural laws through differential equations are not altered, but the field quantities have a meaning as the average of a large number of measurements. For wave motion, the definitions lose their meaning in the region $\lambda \leq \gamma$. If a photon with $\lambda > \gamma$ goes from a frame K to another K′ by a co-ordinate transformation with $\lambda' < \gamma$, the interpretation is that the photon loses its wave character but appears as a static field.

The notion of statistical metric can also be applied to quantum electrodynamics, and March has shown that the Heisenberg-Pauli theory can be developed so as to get rid of all the difficulties connected with infinities, and that too in a relativistic invariant manner. A consequence of the new metric is that in place of Dirac's function an undetermined function $D(x-x')$ appears in the exchange relations. The function D is different from zero inside G (region of dimensions γ), and is transformed on a change of co-ordinates like Dirac's β-matrices. It effects an indeterminacy in orthogonal systems, and hence a breaking off of the spectrum at $\lambda=\gamma$, and the radiation energy will be finite. The proper energy corresponding to terms $i=k$ (which become infinite in the old theory) is here finite and equal to $3e^2/2\gamma$. This expression does not therefore denote any special inner condition on the electron, and the eigen-energy and hence also the mass of the electron are to be considered as magnitudes forced on it by the world geometry. Thus γ and not m is to be considered as a primary natural constant.

The application to the photon gives a finite value for the gravitational energy of the photon which on the old theory leads to infinities as shown by Rosenfeld (48) and Solomon (49). This result has a fundamental significance in the theory because it is here applied to a case where the electron makes no appearance, and shows that γ is not merely a physical constant connected with the electron but an universal natural constant.

Following the ideas of March, Foradori (50) has attempted a revision of the mathematical foundations of the physical space-continuum. The division of the physical continuum into arbitrarily small parts on any fundamental principles makes the geometrical notion of a point, and hence also the "mengentheoretische" point-continuum physically meaningless. These difficulties could be avoided if the space continuum be described not through the connection of unextended points but by the property of divisibility into parts. This is proposed to be done with the aid of the topological "Teiltheorie" which makes continua whose divided parts have a lower limit, understandable mathematically. The basis of this theory is the isomorphy idea, which shows that physical space (represented by its conditioned divisibility) is isomorphous with the geometrical point-continuum (of unlimited divisibility) and this result allows the simple description of physical phenomena by means of a shortest length. The indeterminacy resulting from such smallest volume elements condition a notion of overlapping which leads at once to a statistical metric.

6. Nuclear Physics

A third fundamental difficulty facing quantum mechanics is the theory of the elementary particles—specially the heavy particles which are not dealt with under one comprehensive theory. The specifically nuclear forces have been recently explained with partial success on the basis of the mesotron theory (51) which again meets with difficulties in the region of energies high compared with the rest energy of the mesotron (52). These difficulties have suggested several theories of a geometrical nature.

One of the most striking of these theories is that recently proposed by Born (53) based on what is called the "*principle of reciprocity*". The first idea behind this principle is that, since space-time and energy-momentum appear symmetrically in quantum mechanics, an attempt at the construction of a geometry might be made in the momentum-space just as well as in the co-ordinate space. The second idea is that the geometry is to be of a non-linear type since the non-linearity of the equations of quantum-mechanics seem to survive in all attempts at generalised quantum field theories. The only natural way of introducing non-linearity into a field theory is that of Einstein's theory of gravitation. These ideas have led to the formulation that the domain of the elementary particles has to be considered from the stand point of *momentum space* (since an existence of absolute momentum seems quite plausible) *in which a non-linear geometry with small curvature reigns*, but the curvature equation should contain solely the constants a (of universal length) and b (universal momentum) but not the gravitational constant since gravitation is evidently only a macroscopic effect.

Another purely classical theory to explain the masses of the elementary particles has been proposed by Proca and Goudsmit (54) in which the fundamental idea consists in defining the motion of elementary particles as geodesics of a configuration space carrying besides the ordinary space-time co-ordinates, two others, one corresponding to a charge and the other to spin. Thus the metric of this extended space would be written in the form

$$ds^2 = c^2 dt^2 - \sum (dx^i)^2 + \lambda_s \sum (dx^s)^2 + \lambda_c (dx^c)^2$$

and assumed invariant with respect to Lorentz-transformations. This simple classical theory explains the difference in masses of proton and neutron, furnishes a criterion for distinguishing light and heavy particles, completes the relation between electron and photon on Peroni's theory, and finally explains the mass of the mesotron.

Finally mention might be made of a new "Wave-Geometry" comprehensively worked out by Japanese theoretical physicists (55) in which they claim to have shown that *Geometry is theoretical physics as it stands*, a claim which, if substantiated, would mean the realisation of the dream of complete geometrisation of physics. The basic assumptions of this theory are the assumption that the domain of existence of physical objects is a 4-dimensional manifold, and that this manifold correspondly to the pure analytical frame (in the topological sense.) The linear operator $ds = \gamma_i dx^i$ is introduced with

$$\tfrac{1}{2}(\gamma_i \gamma_j - \gamma_j \gamma_i) = g_{ij}$$

which is obtained by linearising the metric

$$ds^2 = g_{ij} dx^i dx^j$$

With this *operator* ds is combined the wave function ψ, and the metric of the manifold is taken as $ds\psi$. Such transformations and parallel displacements are considered in the theory as make $ds\psi = 0$ invariant. This theory has been worked out at great length and applied to most problems in theoretical physics, but there appears to be one important trouble in as much as a wave geometry cannot be constructed for the electron-itself (56).

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ON DEMLO-NUMBERS

BY

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1. In *The Mathematics Student* Vol. VII, No. 2, some of the properties of Demlo-numbers were considered by D. R. Kaprekar. I shall now prove an interesting result depending upon one of the properties of Demlo-numbers.

In what follows $[x]$ denotes the highest integer in x; $(a)^l$ denotes the digit 'a' repeated 'l' times; and $m_k = \left[\frac{(9)^k}{p}\right] - \left[\frac{(9)^{k-1}}{p}\right]$ where p is an integer. By definition, m_k denotes the numer of all the numbers having k digits and divisible by p.

2. A Demlo number is defined* as an integer which may be partitioned into three groups of which the middle group is of the form $(a)^l$, and the sum of the numbers in the first and the last groups (supposed to have the *same* number of digits) is of the form $(a)^m$; l may even be equal to zero, and the remaining two groups can exist with the above property.

2·1 The following Theorem of D. R. Kaprekar, is used in the sequel:

THEOREM:—*If N is a number with k-digits and P a number consisting of l ($l \geqslant k$) repeated 9's, then $N \times P$ is a Demlo-number in which the first part is $(N-1)$ and the middle part consists of $(l-k)$ repeated nines.*

2·2 It follows that if

$$P = (9)^l,\ N \times (9)^l = (N-1)\ (9)^{l-k}\ \{(9)^k - (N-1)\}.$$

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* The Mathematics Student, Vol. VI p.

Thus $\quad 27583 \times (9)^5 = 2758299972417$

I shall prove, here, the following

THEOREM:—*If a number* $P = N_1\ x_1\ x_2 \cdots x_{l-k}\ N_2\ (l \geq k)$ *of* $l+k$ *digits* (N_1 *and* N_2 *having each* k *digits*) *divided by a number* $p \times (9)^l$ *yields a quotient equal to* $\left[\frac{N_1}{p}\right]$ *with a remainder* $1x_1\ x_2 \ldots \ldots x_{l-k}\ (9)^k$;

then *Case* (i); if p divides neither $(9)^{k-1}$ nor $(9)^k$, the number of such numbers P having $l+k$ digits is precisely m_k;

Case (ii); if p divides both $(9)^{k-1}$ and $(9)^k$ the number of numbers P having $l+k$ digits is precisly m_k;

Case (iii); if p divides $(9)^{k-1}$ but not $(9)^k$ the number of numbers P having $l+k$ digits is (m_k+1);

and *Case* (iv); if p divides $(9)^k$ but not $(9)^{k-1}$ the number of numbers P having $l+k$ digits is (m_k-1);

Proof:—We have

$$N_1\ x_1\ x_2 \ldots \ldots x_{l-k}\ N_2 - 1x_1\ x_2 \ldots \ldots x_{l-k}\ (9)^k$$
$$\equiv p \times (9)^l \times \left[\frac{N_1}{p}\right] = p \times \left[\frac{N_1}{p}\right] \times (9)^l \qquad \ldots (A)$$

The left hand side is equal to

$$(N_1 - 2)\ 9^{l-k}\ (N_2+1), \qquad \ldots (B)$$

and the right hand side is, by (2·1) equal to

$$\left\{p \times \left[\frac{N_1}{p}\right] - 1\right\} 9^{l-k'} \left\{(9)^{k'} - p \times \left[\frac{N_1}{p}\right] + 1\right\} \qquad \ldots (C)$$

where k' is the number of digits in the number $p \times \left[\frac{N_1}{p}\right]$.

It follows then that $k' = k$, and so from (B) and (C)

$$N_1 - 2 = p \times \left[\frac{N_1}{p}\right] - 1$$
$$\therefore \quad N_1 - 1 = p \times \left[\frac{N_1}{p}\right]$$

so that $(N_1 - 1)$ must be a multiple of p.

m_k gives us the number of these $(N_1 - 1)$'s which have k digits in each. But we want the number of numbers N_1 which have k digits in each. Let us consider the four cases separately;

Case (i): Here all the m_k multiples of p are possible values of N_1-1. Therefore the number of numbers of the form $P = N_1\, x_1\, x_2\, x_3 \ldots . x_{l-k}\, N_2$ is m_k;

Case (ii): Since p is a factor of $(9)^k$, one of the (N_1-1)'s is equal to $(9)^k$. But this number does not give us an N_1 having k-digits. Therefore one of the m_k numbers has to be rejected. But, since p divides $(9)^{k-1}$, we get an additional $N_1-1=(9)^{k-1}_3$ which gives a k-digited N_1, and which is not contained in our m_k numbers which are multiples of p. Therefore in this case the number of N_1's is $(m_k-1+1)=m_k$;

Case (iii): Since p divides $(9)^{k-1}$ but not $(9)^k$, from the discussion of the case (ii), we can have the number of N_1's as m_k+1;

Case (iv): A similar discussion, gives the number of N_1's as m_k-1.

3. *Illustration*:—Let us take $l=5$, and $k=3$,

Case (i) Let p be equal to 4 which divides neither 99 nor 999.

Here $l-k=2$. The least $N_1x_1x_2N_2$ is $101\, x_1x_2\, 899$. The greatest is $997x_1x_2003$. Therefore the total number of such N_1's is

$$\frac{997-101}{4}+1=225.$$

By the case (i) of my theorem $m_3 = \left[\frac{999}{4}\right] - \left[\frac{99}{4}\right] = 249-24=225.$

Case (ii) Let p be equal to 3 which is a factor of both 99 and 999. The least $N_1x_1x_2N_2$ is $100x_1x_2900$, and the greatest is $997x_1x_2003$. So the total number of such N_1's is $\frac{997-100}{3}+1=300$. By the case (ii) of my theorem $m_3=\left[\frac{999}{3}\right]-\left[\frac{99}{3}\right]=333-33=300.$

Case (iii) Let p be equal to 11 which is a factor of 99 but not of 999. The least and the greatest $N_1x_1x_2N_2$ are $100x_1x_2900$ and $991x_1x_2009$ respectively. The total number of such N_1's is

$$\frac{991-100}{11}+1=82.$$

By my theorem also, it is $=m_3+1=82$.

Case (iv) Let p be equal to 27 which is a factor of 999 but not of 99,

The least and the greatest $N_1 r_1 r_2 N_3$ are 10999891 and 97399027 respectively. Therefore the total number of such N_1's is

$$\frac{973-109}{27}+1=33.$$

By my Theorem it is $m_3-1=33$.

4. The illustration in case (ii) of § 3 gives the solution to the question 1778 of D. R. Kaprekar.*

Note:—$p\times(9)^5=3\times(9)^5=299997$ is here a 6-digited number.

A NOTE ON THE HARMONIC ENVELOPE

BY

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1. The envelope of lines cutting two conics S, S′ in point pairs separating each other harmonically is a conic Γ called the harmonic envelope of S and S′. We make use of the following well-known properties of the harmonic envelope viz.,

(1·1) The three conics S, S′ and Γ have a common self polar triangle;

(1·2) The eight tangents of S, S′ at their four common points all touch Γ:

(1·3) More generally, the common tangents of pairs of conics like $S+\lambda S'$, $S-\lambda S'$ in the pencil determined by S and S′ are also tangents to the conic Γ.

In this note† we seek to establish some of the relations between the conics S and S′ when Γ degenerates into a pair of points. We first prove the following main theorem viz.,

THEOREM A:—*The necessary and sufficient condition that the harmonic envelope of two conics S, S′ with four distinct intersections may be a point pair is that the sum of the parameters of two of the line pairs in the pencil $S+\lambda S'$ should vanish.*

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* *The Mathematics Student*, Vol. VII, No. 2.

† Theorems C and D of this note have already been obtained by Mr. Robinson; *Vide* "A Note on the Geometric Interpretation etc..." *Bull. Am. Math. Soc.* vol. *XLI*, 1935 pp. 400-402. But theorem A of this note is more fundamental and the synthetic methods used here are more instructive than the verifications in Mr. Robinson's note.

2. Let $S_1 \equiv S - \lambda_1 S'$, $S_2 \equiv S - \lambda_2 S'$, $S_3 \equiv S - \lambda_3 S'$ be the line pairs through the four intersections A, B, C, D of S and S′. Since a line pair as an envelope is the squared point at the intersection of the two lines it follows from (1·3) that the conics $H_1 \equiv S + \lambda_1 S'$, $H_2 \equiv S + \lambda_2 S'$, $H_3 \equiv S + \lambda_3 S'$ have double contact with Γ. In particular when Γ degenerates into a pair of points X, Y by (1·1) X, Y lie on a side of the diagonal triangle of the four points and separate harmonically the diagonal points on that side. Now, if a non-degenerate conic is to have double contact with a conic envelope consisting of a pair of points X, Y it is necessary that the conic should pass through X, Y. As however, only one of the conics H_1, H_2, H_3, say H_1, can pass through XY it is obvious that H_2, H_3 must be degenerate. Thus H_2 is the line pair S_3 and H_3 is the line pair S_2. Hence the parameters of S_2, S_3 form a dyad apolar to the dyad $(0, \infty)$.

From the preceding result we can immediately obtain the relation between the invariants of S, S′ when Γ is a pair of points. In this case the roots of the Lame's equation $\Delta'\lambda^3 + \theta'\lambda^2 + \theta\lambda + \Delta = 0$ are $\lambda_1, \lambda_2, -\lambda_2$; so thar $\lambda_1 = -\frac{\theta'}{\Delta'}$, $-\lambda_2^2 = \frac{\theta}{\Delta'}$ and $-\lambda_1\lambda_2^2 = -\frac{\Delta}{\Delta'}$. Thus the invariant relation† is seen to be $\theta\theta' - \Delta\Delta' = 0$.

From theorem A it follows that if S be a non-degenerate conic of a pencil of conics whose line pairs are S_1, S_2, S_3 the conics S′ in the pencil which are so related to S that the harmonic envelope of S, S′ breaks up into a pair of points are the apolar compliments of S with respect to any two of the line pairs. If, however the line pair S_1 be one of the conics related to S in this way the Lame's equation for the system $S_1 + \lambda S$ viz., $\Delta\lambda^3 + \theta\lambda^2 + \theta_1\lambda + \Delta_1 = 0$ will have for roots $0, \lambda, -\lambda$. Thus $\theta = 0$ i.e., S_1 is out-polar to S so that the four points ABCD form a harmonic set on S. We have thus:

THEOREM B:—*In a pencil of conics with four distinct intersections there are three nondegenerate conics so related to a given nondegenerate conic of the pencil that their harmonic envelope breaks up into a pair of points, unless the base points form a harmonic set on the given conic in which case there are only two.*

3. Let the line pairs AB, CD; AC, BD; AD, BC intersect in P, Q, R respectively and let the harmonic envelope of S, S′ consist of

† *Vide* CASEY: *Treatise on Analytical Geometry*, page 481, example 4.

the pair of points X, Y on PR. By (1·2) it is easily seen that the tangents at two of the base points say A, C to S and at the other base points C, D to S′ pass through one of the points X, Y—say X. Similarly the tangents at A, C to S′ and at B, D to S pass through Y. Let XA, XB meet CD in V, W respectively. Now, the line pairs X (AC, BD, PP) belong to an involution pencil, i.e., VC, WD, PP form an involution range. Therefore, $VD \cdot WP \cdot PC + CW \cdot DP \cdot PV = 0$ from which we have (VPCD) = −(PWCD). But (VPCD) = A (ABCD) on S and (PWCD) = B (ABCD) on S′. We have thus the result:

THEOREM C:—*A necessary and sufficient condition that the harmonic envelope of two conics S, S′ with four distinct intersections may break up into a pair of points is that the cross-ratio of the four points in some order on one of them be the negative of the corresponding cross-ratio on the other.*

Let the tangents at A to S, S′ meet S′ and S in U′ and U respectively By Theorem A, S and S′ are apolar compliments of each other with respect to two of the line pairs, say $S_1 \equiv (AB, CD)$, $S_3 \equiv (AD, BC)$. Hence by (1·3) the harmonic envelope of S_1, S_3 which is a conic touching the lines AB, BC, CD, DA belongs to the range determined by S and S′. Therefore by Sturm's Theorem A (UU′, BD) = −1 i.e., (UA, BD) = −1 on S and (U′A, BD) = −1 on S′. Thus since C is a point on S and S′, C (UA, BD) = −1 and C (U′A, BD) = −1 so that CUU′ are collinear. Similar relations hold good for the points B, C, D. Calling U, U′ the associated points of A, we have:

THEOREM D*:—*If two conics with four distinct intersections be such that their harmonic envelope breaks up into a pair of points, the associated points of each intersection are collinear with another intersection.*

* *vide* Sommer ville. *Analytical conics*, p. 293, example 35.

NOTES AND DISCUSSIONS

Joule's "Minimum Dissipation" Theorem

In all standard treatises on Electrical theory, it is proved that when an electric current distributes itself in a net-work of conductors in accordance with Kirchoff's laws, the expression $\sum C^2R$ is a minimum, C being the current flowing in a typical conductor of resistance R; i.e. if the same total current were distributed in some other way consistent with Kirchoff's first law (viz. continuity of current), then the value of $\sum C^2R$ in this case would be greater than that for the natural distribution determined by Kirchoff's second law (which is an immediate consequence of Ohm's law. Now since the heat generated in a typical conductor in C^2R per second, the result obtained above is interpreted as follows :—

"In any system of conductors in which there are no internal electromotive forces the heat generated by currents distributed in accordance with Ohm's Law is less than if the currents had been distributed in any other manner consistent with the actual conditions of supply and outflow of the current."

The quotation is from *Maxwell's Electricity and Magnetism*, Volume I, section 284, and all authors that we have been able to consult draw the same conclusion. The result is known as "JOULE'S LAW OF MINIMUM DISSIPATION" and it has been accepted without question for well-nigh a century,—probably on account of its analogy with other well-known "minimum" theorems of Physics. It is the purpose of this note, however, to show that the conclusion quoted above is not justified.

The rate of heat-generation in a conductor to which a current C is supplied is CV, V being the total fall of potential in the conductor. This result is independent of Ohm's Law. If Ohm's Law is true, $V=CR$ where R is a constant for the conductor and then the rate of heat-production is C^2R. If, however, Ohm's Law is not true, the rate of heat-production, CV, is not equal to C^2R; and the minimum property of the expresion $\sum C^2R$ (which is proved) is not equivalent to the minimum property of heat-generation (Joule's Law). In fact it is obvious that so long as the total current supplied to the net-work, at a given potential difference, remains unchanged, the rate of heat-generation is independent of the distribution of current, in the various conductors. In other words Joule's Law is not true.

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On the number π

(Dedicated to the shades of my pupil Celina Glass)

We give in this note a brief calculation of the number π to two places of decimals, by the method of Archimedes. This calculation can be understood by pupils of secondary schools.

Denoting by a_n a side of a regular n-gon inscribed in a circle with the radius 1, we have

$$a_6 = 1$$

$$a_{2n} = \sqrt{2 - \sqrt{4 - a_n^2}}$$

To prove the last relation, we have the equations

$$a_n = 2 \sin \pi/n, \quad a_{2n} = 2 \sin \pi/2n,$$

since the radius of the circle is unity.

Therefore
$$\begin{aligned} a_{2n} &= 2\,(\sin^2 \pi/2n)^{\frac{1}{2}} \\ &= 2\sqrt{\tfrac{1}{2}(1 - \cos \pi/n)} \\ &= 2\sqrt{\tfrac{1}{2}(1 - \sqrt{1 - \sin^2 \pi/n})} \\ &= 2\sqrt{\tfrac{1}{2}(1 - \sqrt{1 - a_n^2/4})} \\ &= \sqrt{2 - \sqrt{4 - a_n^2}}. \end{aligned}$$

So, we have
$$\begin{aligned} a_{12} &= \sqrt{2 - \sqrt{3}} \\ a_{24} &= \sqrt{2 - \sqrt{2 + \sqrt{3}}} \\ a_{48} &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{3}}}} \\ a_{96} &= \sqrt{2 - \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{3}}}}} \quad \ldots \; (1) \end{aligned}$$

The side A_{96} of a regular 96-gon circumscribed to this circle we get from formula

$$A_{96} = \frac{a_{96}}{\sqrt{1 - \frac{1}{4} a_{96}^2}}$$

This follows from the fact that

$$\begin{aligned} A_n &= 2 \tan \pi/n \\ &= 2\,\frac{\sin \pi/n}{\cos \pi/n} \\ &= \frac{a_n}{\sqrt{1 - a_n^2/4}}. \end{aligned}$$

From inequalities between the circumferences of these figures

$$96\ a_{96} < 2\pi < 96\ A_{96},$$

we obtain

$$48\ a_{96} < \pi < \frac{48\ a_{96}}{\sqrt{1-\frac{1}{4}\ a_{96}^2}} \qquad \dots \quad (2)$$

We give now values of the roots necessary for the calculation of a_{96} from formula (1). We calculate the upper bounds of the errors as follows ($A>0$, $\alpha>0$):

$$\sqrt{A+\alpha}-\sqrt{A}=\frac{(\sqrt{A+\alpha}-\sqrt{A})(\sqrt{A+\alpha}+\sqrt{A})}{\sqrt{A+\alpha}+\sqrt{A}}=\frac{\alpha}{\sqrt{A+\alpha}+\sqrt{A}}.$$

Thus

$$\sqrt{A+\alpha}-\sqrt{A}<\frac{\alpha}{2\sqrt{A}}.$$

Values of the roots	Upper bounds of the errors
$\sqrt{3}=1{\cdot}732050\ ..$	0
$2+\sqrt{3}=3{\cdot}732050\dots$	0
$2+\sqrt{3}=3{\cdot}732050$	10^{-6}
$\sqrt{2+\sqrt{3}}=\sqrt{3{\cdot}732050}=1{\cdot}931851\ ..$	$\frac{10^{-6}}{2\times 1{\cdot}9\dots}<10^{-6}$
$2+\sqrt{2+\sqrt{3}}=3{\cdot}931851\dots$	10^{-6}
$2+\sqrt{2+\sqrt{3}}=3{\cdot}931851$	$10^{-6}+10^{-6}=2\times 10^{-6}$
$\sqrt{2+\sqrt{2+\sqrt{3}}}=\sqrt{3{\cdot}931851}=1{\cdot}982889\dots$	$\frac{2\times 10^{-6}}{2\times 1{\cdot}9\dots}<10^{-6}$
$2+\sqrt{2+\sqrt{2+\sqrt{3}}}=3\ 982889\dots$	10^{-6}
$2+\sqrt{2+\sqrt{2+\sqrt{3}}}=3{\cdot}982889$	$10^{-6}+10^{-6}=2\times 10^{-6}$
$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}=\sqrt{3{\cdot}982889}=1{\cdot}995717\dots$	$\frac{2\times 10^{-6}}{2\times 1{\cdot}9\dots}<10^{-6}$

Values of the roots	Upper bounds of the errors
$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}} = 1{\cdot}995717$	$10^{-6}+10^{-6}=2\times10^{-6}$
$2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}} = 0{\cdot}004283$	2×10^{-6} (excess)
$a_{96}^2 = 2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}} = 0{\cdot}004281$	2×10^{-6}
$a_{96} = \sqrt{0{\cdot}004281} = 0{\cdot}06542\ldots$	$\frac{2\times10^{-6}}{2\times0{\cdot}05} = 2\times10^{-5}$
$a_{96} = 0{\cdot}06542$	$2\times10^{-5}+10^{-5}=3\times10^{-5}$
$48\ a_{96} = 48\times0{\cdot}06542 = 3{\cdot}14016$	$48\times3\times10^{-5}<200\times10^{-5} = 2\times10^{-3}$

Then

$$3{\cdot}14 < 48\ a_{96} < 3{\cdot}143,$$

$$\frac{48\ a_{96}}{\sqrt{1-\frac{1}{4}\ a_{96}^2}} < \frac{3{\cdot}143}{\sqrt{1-\frac{1}{4}\times0{\cdot}012}} = \frac{3{\cdot}143}{\sqrt{1-0{\cdot}003}} = \frac{3{\cdot}143}{\sqrt{0{\cdot}997}}$$

$$= \frac{3{\cdot}143}{0{\cdot}998\ldots} < \frac{3{\cdot}143}{0{\cdot}998} = 3{\cdot}14\ldots$$

Therefore we have from inequality (2)

$$3{\cdot}14 < \pi < 3{\cdot}14\ldots$$

and finally

$$\pi = 3{\cdot}14$$

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GLEANINGS

The real tragedy about Ramanujan was not his early death. It is of course a disaster that any great man should die young, but a mathematician is often comparatively old at 30 and his death may be less of a catastrophe than it seems. Abel died at 26 and, although he would no doubt have added a great deal more to mathematics, he could hardly have become a greater man. The tragedy of Ramanujan was not that he died young, but that, during his five unfortunate years, his genius was misdirected, side-tracked and to a certain extent distorted.

G. H. HARDY: Lecture on ' The Indian Mathematician Ramanujan " *Amer. Math. Monthly*, 1937 p. 142.

REVIEWS

R. WEYRICH *Die Zylinderfunktionen und ihre Anwendungen:* B. G. Teubner, Leipzig, 1937.

This is a small and compact book of about 140 pages proposing to establish the fundamental properties of cylinder functions starting from Sommerfeld's integral representation of these functions. A knowledge of the theory of functions of a complex variable is presupposed, which enables the author to deduce the fundamental properties of these functions quite simply and elegantly.

The book is divided into twelve sections, in the first two of which the author, starting from a general plane wave function, arrives at the spherical and cylindrical wave functions by the so-called "*Uberlegungsprinzip.*" Thereby a direct approach to the intuitive grasp of the cylinder functions is made possible. This principle leads us to Sommerfeld's integral representation of the two Hankel's functions of zero order, from which the Hankel's functions $\left(H_n^{\mu}, \mu=1, 2\right)$ of arbitrary order, the Bessel's functions and differential equations satisfied by them are deduced. In section **3** the author proves periodicity and other properties of these functions together with their representation in power series. Sections **4** to **7** are devoted respectively to asymptotic expansions of Hankel, Debye and Airy; representation by integrals of Poisson's type; the functional equation satisfied by cylinder functions and the zeros of cylinder functions. The next five sections deal with miscellaneous topics. In **8**, we find some integrals involving cylinderfunctions, while 9 is devoted to Lommel's transformation and to the functions of anger, Weber-Lommel, and Sturve. In **10** addition and multiplication theorems are deduced and in **11** the eigenvalve problem of a special type of Sturm-Liouville equation and the expansion of an arbitrary function by means of Bessel functions are studied. The last section deals with some physical application. Some problems in small oscillations, electromagnetic wave theory and heat conduction are considered.

FLORIN VASILESCO—*La Notion de Capacitè*—Hermann et Cie Paris, 1937 (Actualités Scientifiques et Industrielles) N° 571.

FLORIN VASILESCO—*La Notion de Point Irrègulier*—Hermann et Cie, Paris, 1938 (Actualités Scientifiques et Industrielles) N° 660.

The Dirichlet problem is one of the best illustrations of the apparently paradoxical saying of H. Poincaré that a problem is never solved but only more or less solved, because its so called solution is really a new problem in disguise Thus Riemann believed that he had solved the Dirichlet problem, but his demonstration, which depends on the minimum on an integral raises the question of the existence of such a minimum, and Weierstrass has shown that the minimum is not always attained. After Riemann, mathematicians of mark like Neumann, Schwarz, Ponicaré, Fredholm did admirable research work to solve the same problem. But they were obliged to consider only *certain*

domains. So their results were again setting new problems, in particular the following one: is it always possible to define a solution in the most general case of a domain?

That the answer must be in the negative was proved first by Prof. H. Lebesgue, in 1913. In the meantime, the theories of point sets, and of functions of real variables, had been so considerably improved that it was possible to tackle the Dirichlet problem from an altogether new point of view. A great number of valuable investigations followed the article of Prof. Lebesgue, and little by little a new theory of the Dirichlet problem was constituted. The two fascicles under review (and the fasc. N° 578 on "Les nouvelles méthodes de la Théorie de Potentiel et le Problème de Dirichlet" by Prof. Ch. de la Vallée Poussin should be added) attempt at a systematic exposition of the results so far obtained in that direction.

The most fundamental notions are that of regular, and of irregular points, and that of point sets of nul capacity. The former two are due to Prof. Lebesgue. A point p of the boundary S of a domain D is said to be regular if for any continuous function $f(p)$ defined on S, the solution $F(P)$ of the Dirichlet problem—generalized if necessary in the sense of N. Wiener—yields lim. $F(P)=f(p)$ when P tends to p in D. If, for at least one function, this does not obtain, then the point is said to be irregular. Necessary and sufficient conditions that a point be regular have been proposed successively by different mathematicians. Some will remain very useful in practice; but they were not very satisfactory until the notion of capacity was introduced. It was first proposed by N. Wiener for closed sets, then extended to open sets, when they are bounded, by Prof. de la Vallée Poussin. O. Frostmann had given recently new generalizations. The notion of electrostatic capacity of a conductor is well known and by a passage to the limit can be easily defined for an enumerable set of points. Its extension to bounded sets, closed and open, raised, on the contrary, considerable difficulties. His generalization of the Dirichlet problem provided N. Wiener with a method of extension, since it led him to extend the notion of potential. But the decisive step was the introduction of a Stieltjès-Lebesgue integral to express without any notion of density the potential of any distribution of mass over a set of points. This has been done by Prof. de la Vallée Poussin who could thus generalize the famous method of balayage (sweeping out method) of Poincaré, to general domains. The notion of function of set is there fundamental. O. Frostmann has shown how unnecessary restrictions were fatal as long as the notion of density was kept, for it is only a secondary notion, while mass and potential are two primitive notions. The methods of these authors are analytic.

Proff. Fekete, Polya and Szegö have proved that exactly as in the case of a sphere, the notion of capacity is one which it is possible to define geometrically. In the case of a closed set, in particular, it will be the transfinite diameter of the set.

This notion of capacity of a set has made it immediately possible to investigate the distribution of the irregular points on the boundary of the domain, and the existence of a limit of the harmonic function on the same boundary. In the theory of integration, a fundamental notion is that of nul

sets, i.e. sets of measure equal to zero in the sense of Lebesgue. The theory of the Dirichlet problem calls into play the notion of sets of nul capacity. They are a sub class of sets measurable B (in the sense of Borel) and of measure zero. The boundary of any bounded domain D can be divided into two parts, an "improper" part, always of nul capacity, and its complement, the "reduced" part. A harmonic function bounded in the vicinity of a point of the improper part is harmonic even at these points which are always part or totality of the *interior* boundary. Then the "reduced part" itself can be divided into two parts: a set of nul capacity formed by the irregular points, and the set of the regular points. Both these sets may be dense everywhere. At an irregular point the Green's function does not tend to zero.

Another extremely important application of the notion of capacity has been made to the problem of the uniqueness of a solution of the Dirichlet problem in a general domain. The following theorem has been established by Prof. F. Vasilesco: "It is impossible that two harmonic functions be distinct and bounded in a domain D if their difference tends to zero on the boundary of D except perhaps at the points of an improper set (of nul capacity, therefore).

New results have been obtained also by application of the notion of capacity to the theory of analytic functions. Prof. F. Vasilesco points out some, but a more detailed exposition of this subject will be found in the treatise of R. Nevanlinna "Eindeutige Analytische Funktionen" (Springer 1936).

It would be difficult to summarize in a few words each of these two fascicles, because they are already a summary. In the first one the notion of capacity is the object of the first three chapters. The fourth chapter deals with its generalizations due to O. Frostmann, who has considered generalized potentials. A last chapter reviews the principal properties of sets of nul capacity. The second fascicle treats exclusively of the Dirichlet problem. The first two chapters give a short historical sketch of the theory, and introduce the fundamental notions. The three following chapters give an exposition of the results due to the recent investigations of Prof. de la Vallée Poussin, O. Frostmann, F. Vasilesco, D. Kellog, and M. Riesz, in particular the generalization of the sweeping out method of Poincaré. This is a most admirable application of the integral of Stieltjès-Lebesgue. The sixth chapter gives a clear account of the method of M. Brelot who has treated the inverse problem, making use of the properties of subharmonic functions. The last chapter is about analytic functions.

That Prof. F. Vasilesco is one of the best specialists to explain a theory where so many very delicate notions and so many results have to be referred to, is obvious. But to be one of the best specialists in a question does not necessarily mean that one will be clear, chiefly when conciseness is required. Yet as to limpidity, precision and, at the same time, erudition, few tracts of the same series can compete with these two. They are really models of elegant exposition.

C. RACINE.

S. MANDELBROJT "*La Régularisation des Fonctions*" (Actualités Scientifiques et Industrielles, No. 733, Price 11 fr.)

Let $f(z) = \Sigma\, a_n z^n$, $g(z) = \Sigma\, b_n z^n$ be two integral functions. Let $A_n = |\, a_n \,|$ $B_n = |\, b_n \,|$ and let A (r) and B (r) denote the maximum terms in the sequences $\{A_n r^n\}$ and $\{B_n r^n\}$ for a given r. If $A_n \geq B_n$ for all large values of n it is easily shown that A $(r) \geq$ B (r) for all large values of r. But if A $(r) \geq$ B (r) for all large values of *r it is not true* that $A_n \geq B_n$ for all large n. Suppose we plot the points $(n, \log A_n)$. Let $\{\bar{A}_n\}$ be the sequence such that $(n, \log \bar{A}_n)$ is on the side of the least convex polygon enclosing the points $(n, \log A_n)$. Let $\{\bar{B}_n\}$ denote the points similarly associated with $\{B_n\}$. Then it can be shown that $A(r) = \max\limits_{n\geq 0} \bar{A}_n r^n$, $B(r) = \max\limits_{n\geq 0} \bar{B}_n r^n$ and that $A(r) \geq B(r)$ *for all large r implies that* $\bar{A}_n \geq \bar{B}_n$ *for all large n*. The effect of replacing $\{A_n\}$ by $\{\bar{A}_n\}$ is, so to say, to smoothen out the irregularities of the sequence $\{A_n\}$. The booklet under review develops a general process by which any function satisfying certain postulates can be replaced by a "regularised" function for which converse theorems of the type mentioned above hold good. The class of functions which can be regularised includes any sequence as a special case. The general method of regularisation is developed in the first chapter while in the second chapter it is shown that converse theorems of the type described above are true for these regularised functions Some applications to sequences in connection with integral functions and the theory of quasi-analytic functions are considered towards the end of the book. The idea of regularising a sequence or a function is important since it has applications to several problems in the theory of functions and the booklet under review provides a general method to effect such a regularisation in any given case.

V. GANAPATHY IYER.

BOOK RECEIVED FOR REVIEW

B. R. SETH : *Two dimensional potential problems connected with rectilinear boundaries,* Lucknow University Studies XIII, 1939, pp. 124.

ANNOUNCEMENTS AND NEWS

Mohammad Khaja Mohiuddin Esq., M.A., Lecturer in mathematics, Osmania University has been elected a member of the Society.

Prof. D. D. Kosambi of the Fergusson College, Poona has been elected a member of the Managing Committee of the Indian Mathematical Society in place of Dr S. Chowla who has resigned.

We offer our hearty felicitations to Dr. K. S. Krishnan of the Indian Association for the Cultivation of Science, Calcutta, who has been recently elected to a Fellowship of the Royal Society. The late S. Ramanujan was the first Indian to be chosen for this signal distinction (at the age of 30). The other Indians to secure a like honour so far are Dr. J. C. Bose, Sir C. V. Raman, Dr. M. N. Saha and Dr. Birbal Sahni.

MATHEMATICAL REVIEWS.—English reading lovers of mathematics will rejoice over this periodical sponsored by the American Mathematical Society and the Mathematical Association of America, which publishes critical reviews of current periodicals and of books. The earliest periodicals to meet this need were the *Jahrbuch uber die Fortschritte der mathematik*, founded in 1871 and the *Revue Semestrielle des publications mathematiques* established about the end of the last century, which became absorbed into the former in 1932. From 1931 the *Zentralblatt fur Mathematik* (Julius Springer) has been offering a somewhat quicker service. but the English reader has been all along left in the cold.

Mathematical Reviews which is printed in large size double column pages appears almost monthly and is at present edited by Prof. J. D. Tamarkin of the Brown University and Prof. O. Neugebauer, who was formerly editor of the *Zentralblatt*. It is priced much below its cost price at $13 per year, with a reduction to $6 5 for members of the sponsoring associations and $3 25 to reviewers. This has been rendered possible by the munificence of the Carnegie Corporation and the Rockfeller Foundation. The Brown University which has perhaps the finest mathematical library in the world is housing the scheme and aiding the editorial work.

An ambitious scheme possible only in the Land of Big Ideas and Long Purses is the micro-film service which is offered to subscribers of The Mathematical Reviews On request from a subscriber the organizers will send him either micro-film photographs at one 35 mm. film per double page (requiring some reading machine or projector) or photo-prints on sheets 11″ × 14″ readable directly, at 2 cents and 16 cents respectively.

The International Congress of Mathematicians which was planned to meet in Cambridge, Mass, U. S. A. in September 1940 has been "postponed until a more favourable time." All membership fees paid so far have been returned.

The authorities of the Madras University have declared Mr. A. Narasinga Rao qualified for the degree of Doctor of Science. The subject of the thesis was "Studies in Circle Geometry."

SOLUTIONS TO QUESTIONS

Question 1694

(V. THEBAULT, Le Mans, France):—With the figures x, $x+1$, $x+2$, $x+3$, $x+4$ form a number whose square contains the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 each taken once.

Solution by the proposer

Let $n = abcde$ be an integer of 5 digits selected from the 5 numbers x, $x+1$, $x+2$, $x+3$, $x+4$. Since n^2 must be a number between 10^8 and 10^9-1 we have

$$12435 \leqslant n < 31624.$$

Hence the first digit in n must be 1, 2 or 3. So

$$1+2+3+4+5=15 \leqslant a+b+c+d+e \leqslant 3+4+5+6+7=25.$$

Also n^2 is a number with digits 1 to 9 taken once. Hence it is divisible by 9. So n is divisible by 3. Therefore

$$a+b+c+d+e=5x+10=5\ (x+2)$$

is divisible by 3 as well as 5. Hence $a+b+c+d+e=15$. And n is a number whose digits are selected from 1, 2, 3, 4, 5. Considering those beginning with $a=1$, there are 24 numbers of which only the number $n=12543$ for which $n^2=157326849$ satisfies the required condition. Similarly considering those beginning with 2, 3, 4 or 5 we find that none of these satisfy the required condition. Hence $n=12543$ is the only solution.

Also solved by N. K. Narasimhamurty.

Question 1708

(V. THEBAULT, Le Mans, France): With the figures 1, 2, 3, 4, 5, 6, 7, 8, 9 taken each once, form the least and biggest numbers which are perfect squares.

Solution by the proposer

Any number whose square consists of the nine non-repeated digits 1, 2, ..., 9 is of the form $N = abcde$. Since the sum of the digits in N^2 is 45, it follows that N^2 is divisible by 9. So N is divisible by 3. Hence $a+b+c+d+e$ is divisible by 3. It is seen by calculation that $10000 < N < 316263$.

To find the least N, we note that $a=1$ in this case. Next $b=0$ or 1. Now $b \neq 0$ for, if $b=0$, we have $(10cde)^2=(10^4+cde)^2$ $=10^8+2\cdot 10^4\cdot cde+(cde)^2$. If $cde \leqslant 499$, $2\cdot 10^4\cdot cde<10^7$ and N^2 contains a zero digit. If $N=500$, N^2 contains 4 zero digits and if $N \geqslant 501$, $2\cdot 10^4\cdot cde>10^7$ and N contains two repeated digits 11 in the first two places. So, $b=1$. Hence the number is of the form $11cde$.

Now, $e \neq 0$ since N^2 will then end in 00. Also $e \neq 1$ or 9 since N^2 will end in 1 and since N^2 begins with 1, it will contain 1 twice. So $e=2, 3, \cdots, 7$ or 8. Using the fact that $1+1+c+d+e$ is divisible by 3, we get that $c+d+e$ must be one of the numbers 4, 7, 10, $\cdots$, 22 or 25. Considering the case $e=2$, we find that $c+d$ must be one of the numbers 2, 5, 8, $\cdots$, 20 or 23. So cd must be of the form 02, 20, 11, 05, 50, 14, 41, 23, 32, 08, 80, etc. It will be seen that none of these are suitable since the square N^2 will contain repeated digits. Similarly the other values of e can be dealt with and it will be found that $cd=82$ is the *only suitable combination.* So the least N is 11826.

A similar procedure will give that the greatest N is 30384. So the least required number is $N^2=139854276$ and the greatest is $N^2=923187456$.

Question 1713

(FRANCIS REGAN, ST. Louis U. S. A.):—Prove that if through any point V on a conic pairs of lines are drawn making supplementary angles with a fixed straight line in the plane, the envelope of the chords which join the intersections of these pairs of lines with the conic is, in general, a hyperbola.

Remarks by T. K. Raghavachari.

The result contained in the question is not correct.

Taking V for the origin and the lines through V which are parallel and perpendicular to the given line L (say) for axes, the equation of any conic S (not passing through V) may be written as $ax^2+2hxy+by^2+2gx+2fy+c=0$. Now if the line $lx+my+n=0$ meets the conic in P, Q so that VP, VQ make supplementary angles with L, it is easily seen that the line envelopes the conic whose tangential equation is $hn^2-gmn-fnl+clm=0$. In particular when $h=0$ i.e., when one of the axes of the conic is parallel to L, the envelope is a parabola and when $c=0$ i.e., when the conic passes through V, the envelope breaks up into the point V and another point.

Taking the case when V lies on the conic, let the lines through V perpendicular and parallel to L meet the conic S in O, O′ and let T be the pole of OO′ w.r.t. the conic. We notice that:

(i) VO, VO′ are the bisectors of the angles between VT and the tangent at V to the conic.

(ii) When L is parallel to an axis of S, OO′ is a diameter.

(iii) If L turns round a fixed point K, OO′ turns round F the Fregier point of V.

QUESTIONS FOR SOLUTION

1785. (V. THEBAULT, Le Mans, France): Find three numbers A, B, C each of 4 digits such that the 16 digit number AABB may be the square of CC.

[Example: 1976 1976 8025 8025 = 4445 4445²]

1786. K. S. K. IYENGAR. Given a class $c(f)$ of functions $f(x)$ defined in $0 \leqslant x \leqslant 2\pi$ with the following properties

(1) f is L^2 in $(0, 2\pi)$

(2) the Fourier series of f converge in the mean to f uniformly in the class $c(f)$.

(3) If f_1 and f_2 are any two functions of $c(f)$ then

$$\int_0^{2\pi} (f_1 - f_2)^2 \, dt \geqslant k \text{ where } k \text{ is a positive constant;}$$

prove that

(A) (1) the class $c(f)$ is enumerable

(2) If the sequence $\{f_n\} \equiv c(f)$ and $R_n = \left\{ \int_0^{2\pi} f_n^2 \, dt \right\}^{\frac{1}{2}}$

then the index of convergence of R_n is finite;

(B) If the class c consists of trigonometric polynomials of degree not greater than n_0 and satisfy condition (3) given in the beginning then the index of convergence of the corresponding R_n's is not greater than $2n_0 + 1$;

(C) A theorem similar to (A) and (B) can be emunciated with regard to algebraic polynomials, the index of convergence being not greater than $n_0 + 1$

1787. (M. N. KRISHNASWAMI): If k integrals

$$f_r\ (x_1 \cdots x_n\ x) = g_r (r=1,\ 2, \cdots k)$$

of the system

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \cdots = \frac{dx_n}{X_n} = \frac{dx}{X}$$

are known, show that a last multiplier of the reduced system is given by $M \div \dfrac{\partial(f_1 \cdots f_k)}{\partial(x_1 \cdots x_k)}$. Examine the case when n integrals of the system are known.

GLEANINGS

Nature of Modern Mathematics

The feature of the modern age which strikes the eye at first sight is its wide and varied use of machinery and the harnessing of natural forces to the service of human needs. That which has made this possible is the mathematical outlook which from the crude beginnings of early human history has now developed into a self-confident full-dress view of the nature of external Reality, founded on the concept of Natural Law.

Philosophical investigations into the nature and method of mathematical science have revealed that Mathematics is essentially a science of relations, and is founded in the last resort on the notion of serial order. The two working concepts of Mathematics—Number and function—are, for instance, purely relational nature, even the 'Numbers,' which are the entities with which Mathematics works, are empty of all concrete content; their being is exhausted in the order relations which define them. In short, Mathematics thinks away the intrinsic essence of the entities with which it deals, there remaining no longer their being-for-itself, but only their relational being or being-for-others. This fact is expressed strikingly in Russell's paradoxical definition, 'Mathematics is the science in which we do not know what we are talking about, nor whether what we say is true.'

The typical being-for-itself is revealed in the notion of the self which arises in self-consciousness. The investigation of the nature of the self which arises in self-consciousness. The investigation of the nature and possibilities of the self through suitable systems of sadhana leads to higher levels of spiritual consciousness and to the experience of God as the upper limit of Being—transcendent and super-relational, or immanent and containing all relations within itself. It follows that the knowledge-effort in man can have two orientations—either towards intrinsic or individual being culminating in its upper limit God, or towards merely relational being or being at its inferior limit in Mathematics. God and Mathematics are thus the extreme poles between which all cognitive experience oscillates; the one typifies the supremely individual being and freedom, the other extrinsic being and mechanism.

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